

Introduction

Let $f \in \mathcal{S}_{k+2}(\Gamma_0(pN))^{\text{new}}$ be a newform, where $k \geq 0$ is even, p is prime and N is an integer not divisible by p . Let χ be a Dirichlet character of conductor prime to pN such that $U_p f = \chi(p)p^{\frac{k}{2}} f$. By work of Mazur, Tate and Teitelbaum the order of vanishing of the p -adic L -function attached to f at the central point $s = \frac{k+2}{2}$ is one higher than that of the classical L -function attached to it. Moreover, they formulated the following **exceptional zero conjecture**: There exists an invariant $\mathcal{L}_p(f) \in \mathbb{C}_p$, depending only on the local Galois representation $\sigma_p(f)$ attached to f , such that

$$L'_p(f, \chi, \frac{k+2}{2}) = \mathcal{L}_p(f) \cdot L^{\text{alg}}(f, \chi, \frac{k+2}{2}).$$

By work of Greenberg-Stevens and many others, the qualitative behaviour of $\mathcal{L}_p(f)$ is well known, on the contrary, not a lot of quantitative data on $\mathcal{L}_p(f)$ for arbitrary even weight is available. For this purpose, our aim is to compute the \mathcal{L} -invariant defined by Teitelbaum in [5].

Objectives

- Find an **efficient method** for computing \mathcal{L} -invariants for modular forms of arbitrary even weight.
- Analyze the **distribution and behaviour** of the \mathcal{L} -invariants for growing weight.

Computing Teitelbaum's \mathcal{L} -operator

Teitelbaum's invariants $\mathcal{L}_p(f)$ are realized as the eigenvalues of an operator, called the **\mathcal{L} -operator**, defined on the finite-dimensional \mathbb{C}_p -vector space of harmonic cocycles on the **Bruhat-Tits tree** \mathcal{T} . There are three main difficulties in designing an efficient method to compute this operator (up to a prescribed precision) as a matrix with p -adic entries:

- The \mathcal{L} -operator in [5] is defined over \mathbb{C}_p . In order to keep the running time of our computations as low as possible, we show that the \mathcal{L} -operator can in fact be **defined over \mathbb{Q}_p** .
- In order to describe Γ -invariant harmonic cocycles on \mathcal{T} by a finite amount of data, it is necessary to compute a **fundamental domain for the action of Γ on \mathcal{T}** , see [1].
- Coleman integrals enter into the definition of the \mathcal{L} -operator following [5] and a priori efficient computation of these seems to be completely out of reach. Teitelbaum proved in [5] that one can replace these Coleman integrals by p -adic integrals coming from harmonic cocycles. However, computing the integrals directly in terms of this new definition is much too slow to compute the \mathcal{L} -operator efficiently. An alternate approach was presented by Greenberg in [3] building on the **overconvergent methods** developed by Darmon, Pollack and Stevens.

To make the overconvergent method applicable in our setting, we prove a **control theorem for p -adic automorphic forms** of arbitrary even weight generalizing [3, Corollary 2].

Quotients of the Bruhat-Tits-tree

- Let B be a definite rational quaternion algebra of discriminant N , that is split at p .
- Let R be a maximal order in B .
- Let $\Gamma = R_{\neq 1}^{\times}$ denote the units of reduced norm 1.
- Fix a splitting $\iota: B \otimes_{\mathbb{Q}} \mathbb{Q}_p \rightarrow M_2(\mathbb{Q}_p)$ such that $\iota(R \otimes_{\mathbb{Z}} \mathbb{Z}_p) = M_2(\mathbb{Z}_p)$ and regard Γ as a (discrete and cocompact) subgroup of $SL_2(\mathbb{Q}_p)$ via the splitting.

The **Bruhat-Tits tree** \mathcal{T} for $GL_2(\mathbb{Q}_p)$ is the graph whose vertices \mathcal{T}_0 are the homothety classes of \mathbb{Z}_p -lattices in \mathbb{Q}_p^2 . Two vertices v and w are joined by an edge in \mathcal{T}_1 if there exist representative lattices L and L' such that $pL \subsetneq L' \subsetneq L$. The graph \mathcal{T} is a $p+1$ -regular tree. Via the reduction map, it encodes the geometry of the p -adic upper half plane $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$. The group $GL_2(\mathbb{Q}_p)$ acts transitively on \mathcal{T} . The quotient $\Gamma \backslash \mathcal{T}$ is a finite graph. A **fundamental domain** for the action of Γ on the tree \mathcal{T} can be efficiently computed, see [1].

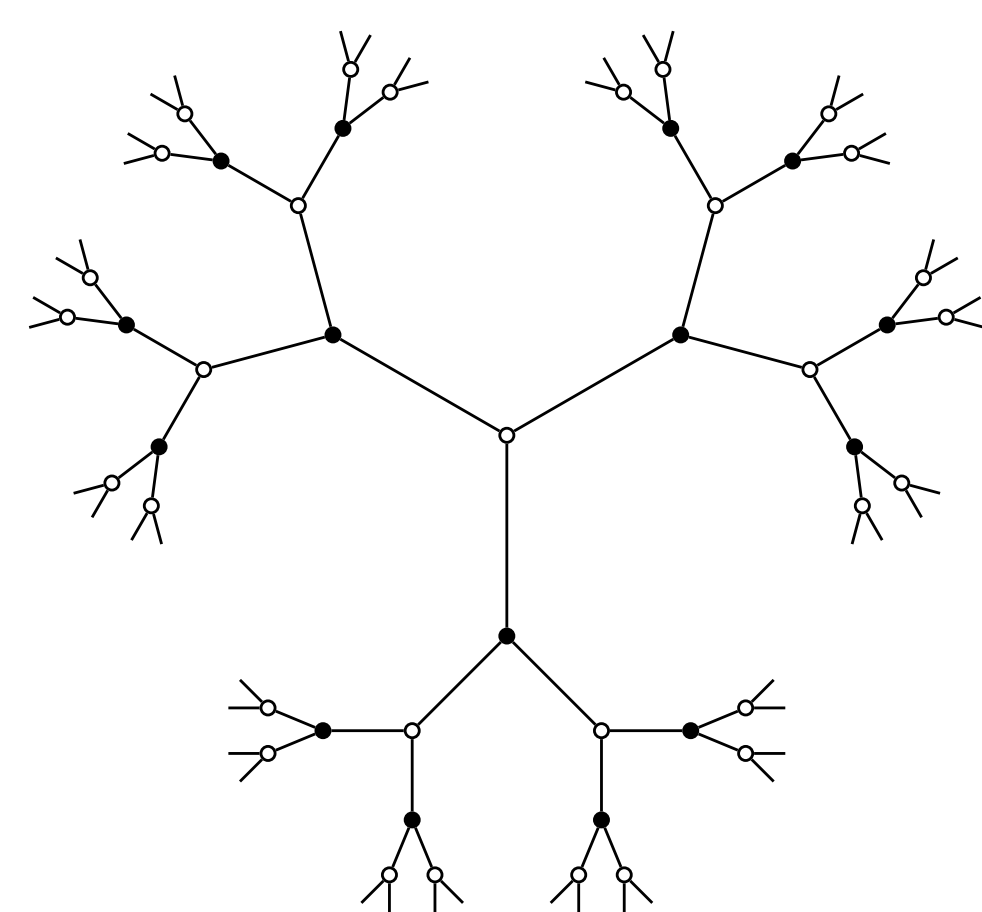


Figure 1: Bruhat-Tits tree for $p = 2$

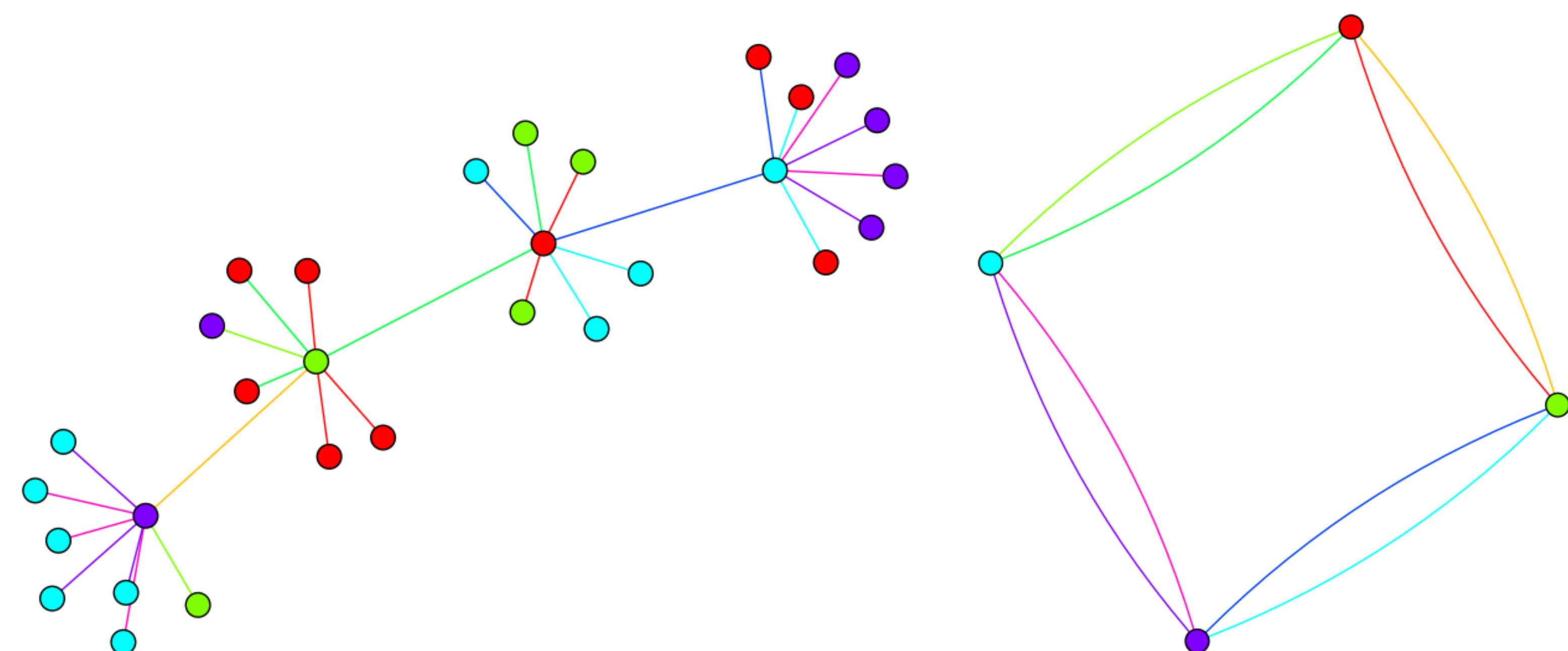


Figure 2: Fundamental domain and quotient graph for $p = 7$ and $N = 11$

The crucial step is to reduce the problem whether two edges (or vertices) are Γ -equivalent to a shortest vector search in a lattice by only using an approximation of ι up to some finite p -adic precision.

Teitelbaum's \mathcal{L} -operator

Let $P_k = \mathbb{Q}_p[x]_{\leq k}$ with the $GL_2(\mathbb{Q}_p)$ -action given by

$$(P \cdot g)(x) = \det(g)^{-\frac{k}{2}} (cx + d)^k P \left(\frac{ax + b}{cx + d} \right), \quad \text{for } P \in \mathcal{P}_k, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$$

and denote by V_k its dual. The space of Γ -invariant harmonic cocycles $C_h(\Gamma, k)$ on \mathcal{T} consists of maps $c: \mathcal{T}_1 \rightarrow V_k$ such that for all $v \in \mathcal{T}_0, e \in \mathcal{T}_1$ and $\gamma \in \Gamma$ one has

$$c(\bar{e}) = -c(e), \quad \sum_{s(e)=v} c(e) = 0, \quad (\gamma \cdot c)(e) = c(\gamma e).$$

There is a Hecke-equivariant isomorphism $\mathcal{S}_{k+2}(\Gamma_0(pN), \mathbb{C}_p)^{\text{new}} \cong C_h(\Gamma, k) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Moreover, any $c \in C_h(\Gamma, k)$ gives rise to \mathbb{Q}_p -valued distribution μ_c on certain locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$. For $\tau \in \mathbb{Q}_p^2 \setminus \mathbb{Q}_p$ define $\kappa_{\text{col}}^{\tau}(c): \Gamma \rightarrow V_k$ by

$$\kappa_{\text{col}}^{\tau}(c)(\gamma)(P) = \frac{1}{2} \text{Tr}_{\mathbb{Q}_p^2/\mathbb{Q}_p} \left(\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x) \log_p \left(\frac{x - \gamma\tau}{x - \tau} \right) d\mu_c(x) \right) \in \mathbb{Q}_p, \quad \text{for } \gamma \in \Gamma, P \in \mathcal{P}_k.$$

This induces the Coleman integration map $\kappa_{\text{col}}: C_h(\Gamma, k) \rightarrow H^1(\Gamma, V_k)$. There is also a combinatorial Hecke-equivariant isomorphism $\kappa_{\text{sch}}: C_h(\Gamma, k) \rightarrow H^1(\Gamma, V_k)$ due to Schneider. The composition

$$\mathcal{L} = \kappa_{\text{col}} \circ (\kappa_{\text{sch}})^{-1}: H^1(\Gamma, V_k) \rightarrow H^1(\Gamma, V_k)$$

is **Teitelbaum's \mathcal{L} -operator**, whose eigenvalues are the \mathcal{L} -invariants of the associated newforms.

A control theorem for p -adic automorphic forms

To compute the Coleman integrals, we first construct a covering of $\mathbb{P}^1(\mathbb{Q}_p)$ given by edges in \mathcal{T} such that the integrands have a nice expression. Then the remaining step is to compute the moments $m(\mu_c, g, i) = \mu_c(g\mathbb{Z}_p)(x^i \cdot g^{-1})$ for all $g \in GL_2(\mathbb{Q}_p), i \geq 0$. These moments are encoded in values of rigid analytic automorphic forms for Γ . Let $\mathcal{A}_k(\Gamma)$ denote the vector space of these forms and $\mathbb{A}_k(\Gamma)$ the space of **p -adic automorphic forms**. We have $C_h(\Gamma, k) \cong (\mathbb{A}_k(\Gamma))^{p-\text{new}} U_{p=p^{k/2}}$. By methods along the lines of [4], we obtain the following **control theorem**:

- The restriction of the specialization map $\rho: \mathcal{A}_k(\Gamma) U_{p=p^{k/2}} \rightarrow \mathbb{A}_k(\Gamma) U_{p=p^{k/2}}$ is an isomorphism.
- Let φ_c the p -adic automorphic form corresponding to c and denote the above lift by Φ_c . Then

$$\Phi_c(g)(x^i) = m(\mu_c, g, i).$$

- For any lift $\Phi_0 \in \mathcal{A}_k(\Gamma)$ of φ_c defined over $\mathbb{Z}_p, n \geq 1$ and $i \in \{1, \dots, n\}$, we have

$$((p^{-k/2} U_p)^n \Phi_0)(g)(x^{k+i}) = m(\mu_c, g, k+i) \pmod{p^{n-i+1}}.$$

Computational Results for $p = 2$

Let $d_k(p, N) = \dim_{\mathbb{C}} \mathcal{S}_k(\Gamma_0(pN))^{\text{new}}$ and denote by $\alpha_{\mathcal{L}}(k, p, N)$ the slopes of the \mathcal{L} -operator. Let $\alpha_{\mathcal{L}}^{\pm}(k, p, N)$ be the slopes with respect to the Atkin-Lehner involution at N .

k	$d_k(2, 3)$	$\alpha_{\mathcal{L}}(k, 2, 3)$	
		$\alpha_{\mathcal{L}}^+(k, 2, 3)$	$\alpha_{\mathcal{L}}^-(k, 2, 3)$
6	1	0 ₁	
8	1		-1 ₁
10	1		0 ₁
12	3	-1 ₁	-4 ₂
14	1	-1 ₁	
16	3	-4 ₂	-2 ₁
18	3	-4 ₂	-1 ₁
20	3	-2 ₁	-6 ₂
22	3	-2 ₁	-4 ₂
24	5	-6 ₂	-2 ₁ , -7 ₂
26	3	-6 ₂	-2 ₁
28	5	-2 ₁ , -7 ₂	-5 ₂
30	5	-2 ₁ , -7 ₂	-6 ₂
32	5	-5 ₂	-3 ₁ , -7 ₂

Table 1: $p = 2, N = 3$

k	$d_k(2, 5)$	$\alpha_{\mathcal{L}}(k, 2, 5)$	
		$\alpha_{\mathcal{L}}^+(k, 2, 5)$	$\alpha_{\mathcal{L}}^-(k, 2, 5)$
6	3	-2 ₂	0 ₁
8	1		-1 ₁
10	3	-5 ₂	0 ₁
12	5	-2 ₂	-1 ₁ , -4 ₂
14	3	-3 ₂	-1 ₁
16	5	-5 ₂	-2 ₁ , -4 ₂
18	7	-5 ₄	-1 ₁ , -4 ₂
20	5	-3 ₂	-2 ₁ , -6 ₂
22	7	-1 ₂ , -8 ₂	-2 ₁ , -4 ₂
24	9	-5 ₄	-2 ₁ , -6 ₂ , -7 ₂
26	7	-1 ₁ , -2 ₄	-2 ₁ , -6 ₂
28	9	-1 ₂ , -8 ₂	-2 ₁ , -5 ₂ , -7 ₂
30	11	-4 ₂ , -7 ₄	-2 ₁ , -6 ₂ , -7 ₂
32	9	-1 ₁ , -2 ₄	-3 ₁ , -5 ₂ , -7 ₂

Table 2: $p = 2, N = 5$

Conjectures

- For $k \in 4\mathbb{Z}, k \geq 8$, we have
- $$\alpha_{\mathcal{L}}^-(k, 2, 3) = \alpha_{\mathcal{L}}^+(k+4, 2, 3) = \alpha_{\mathcal{L}}^+(k+6, 2, 3) = \alpha_{\mathcal{L}}^-(k+10, 2, 3)$$
- and consequently $\alpha_{\mathcal{L}}(k, 2, 3) = \alpha_{\mathcal{L}}(k+6, 2, 3)$.
- For $k \geq 6$ we have $\alpha_{\mathcal{L}}^-(k, 2, 5) = \alpha_{\mathcal{L}}(k, 2, 3)$. Moreover, for $k \in 2+4\mathbb{Z}, k \geq 6$, we have
- $$\alpha_{\mathcal{L}}^+(k, 2, 5) = \alpha_{\mathcal{L}}^+(k+6, 2, 5).$$

Future Directions

- The above conjectures show only a small part of the visible patterns and observations. We are gathering more data for other primes, and are computing \mathcal{L} -invariants of classical and p -adic modular forms building on work of Lauder.
- Our observations are related to recent conjectures of Bergdall and Pollack on the size of the Coleman family through f and its relation with the \mathcal{L} -invariant.

References

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