GAUS-AG on *Rigid meromorphic cocycles*

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In this GAUS-AG we want to understand recent developments in the theory of *singular moduli for real quadratic fields* introduced in [DV21] a theory that was developed in search for an analogy for real quadratic fields of the classical theory of complex multiplication.

The theory of elliptic curves with complex multiplication has spectacular arithmetic applications. Kronecker tackled the problem of constructing abelian extensions of imaginary quadratic fields, known as the *Kronecker Jugendtraum*, via singular moduli, i.e., values of modular functions at imaginary quadratic points of the complex upper half plane $\mathcal{H}_\infty$. Another problem for which in current approaches CM theory is an essential tool is the Birch and Swinnerton-Dyer (BSD) Conjecture. CM theory enters as it allows the systematic construction of global points on elliptic curves via the theory of Heegner points. Given the importance of CM theory to number theory it is natural to ask if a similar approach works for other number fields and real quadratic fields seem like a sensible next goal. Explicit class field theory for real quadratic fields analogous to Kronecker's Jugendtraum has seen a lot of progress recently, culminating in the work of Dasgupta and Kakde [DK23]. Moreover Darmon and Vonk have developed an approach that mirrors several aspects of the theory of singular moduli. This is the approach that we will study in this GAUS-AG. We briefly describe it here.

First note that there is no direct analogue of singular moduli, in the sense of evaluating modular functions at real quadratic points (or RM, by analogy with CM points) of $\mathcal{H}_\infty$; simply because RM points lie on the real line. One idea to find an appropriate analogue of CM theory is to replace $\mathcal{H}_\infty$ by the Drinfeld p-adic upper half plane. This is the point of view proposed by Darmon and Vonk in their theory. For a finite prime p, the Drinfeld p-adic upper half plane is the rigid analytic space $\mathcal{H}_p$ whose $\mathbb{C}_p$-valued points are $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$. It contains real multiplication (RM) points over real quadratic fields K in which p is ramified or inert. Darmon and Vonk managed to find analogues of singular moduli in this setting, the so called rigid meromorphic cocycles from the title. To define them let $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ be the so called *Ihara group* and let $\mathcal{M}$ be the ring of rigid meromorphic functions on $\mathcal{H}_p$. A rigid meromorphic cocycle is an element of $H^1(\Gamma, \mathcal{M}^\times)$. In [DV21], Darmon and Vonk initiated the study of these objects and showed that these classes (or more precisely, the quasi-parabolic classes) can be meaningfully evaluated at RM points. They conjectured that these values lie in composita of abelian extensions of real quadratic fields, making them suitable candidates for an analogue of singular moduli in the real quadratic setting.

In the last 50 years, automorphic methods have entered the theory of complex multiplication, pioneered by the work of Gross and Zagier on the factorisation of differences of singular moduli [GZ85]. The crucial idea there is to relate so called generating series that are constructed out of CM cycles on Shimura curves to derivatives of real analytic families of Eisenstein series. This approach led to the proof of many cases of the BSD Conjecture for a modular elliptic curve in analytic rank 1 in [GZ86]. Moreover it led to a large research program, the Kudla program, that aims at investigating relations of generating series of special cycles on Shimura varieties and Fourier expansions of automorphic forms.

For real quadratic fields one can attempt something similar to the approach of Gross and Zagier. One can package RM cycles on $\mathcal{H}_p$ into modular generating series and try to relate these generating series to derivatives of p-adic (!) families of modular forms. This has been explored by Darmon, Pozzi and Vonk in [DPV21] and [DPV23]. In the archimedean setting, only real analytic Eisenstein series admit variations in families. By contrast, the work of Hida, and its generalisation via the theory of eigenvarieties introduced by Coleman–Mazur, show that the p-adic setting is much richer. A useful feature of the p-adic theory is that cusp forms vary in p-adic families. In addition, their corresponding Galois representations can be interpolated. This additional flexibility of the p-adic setting has been used by Darmon–Pozzi–Vonk to prove some instances of the expected algebraicity results for the analogues of elliptic units for real quadratic fields.

Just like in the classical setting, one can try to generalize the theory and go beyond $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$. In the very recent work [DGL23], Darmon, Gehrmann and Lipnowski have defined rigid meromorphic cocycles for orthogonal groups. Using this they moreover propose a conjectural framework for extending some of the arithmetic theory in the Kudla program for orthogonal groups to the case beyond Shimura varieties, i.e., to orthogonal groups of arbitrary real signature.

The seminar is organized as follows. After recalling the theory of complex multiplication we will introduce and study the concept of rigid meromorphic cocycles following [DV21]. We will then discuss the connection to the theory of p-adic families of modular forms as explored in [DPV21, DPV23]. We end by discussing the generalization of the concept of rigid meromorphic cocycles to orthogonal groups, [DGL23].

The seminar takes place weekly on Fridays 9:15-10:45 in hybrid format via Zoom (please contact the organizers for the zoom details) and in room SR 8 of the Mathematikon in Heidelberg. The first meeting is on October 20th. There is no talk on December 8th due to the Ruth Moufang Lectures. Talk 5 has been rescheduled form Friday to Thursday. The seminar will end on February 9th.

Talk 1: Introduction and overview (20.10., Judith Ludwig)

Talk 2: Overview of the theory of complex multiplication (27.10., Marius Leonhardt)
This talk should give an overview of the theory of complex multiplication. It should also include a discussion of "Heegner constructions", i.e., singular moduli, elliptic units and Heegner points. References: [FLPSW23, §2] and the references therein.

Talk 3: Factorization of singular moduli (03.11., Sriram Chinthalagiri Venkata)
This talk should discuss the seminal works [GZ85] and [GZ86] of Gross and Zagier. A special focus should lie on [GZ85, Theorem 1.3] and its analytic proof. References: [GZ85], [GZ86] and [FLPSW23, §4.1].

Talk 4: Rigid meromorphic cocycles (10.11., Janne Frenz)
This talk introduces the protagonists of our seminar, the rigid meromorphic cocycles. For that the talk should start with a brief discussion on Drinfeld p-adic upper half plane $\mathcal{H}_p$ as well as its real multiplication (RM) points. Then introduce rigid meromorphic cocyles (and variants) and give some example. Explain that the rigid meromorphic cocycles $H^1_f(\Gamma, \mathcal{M}^\times)$ can be understood through an additive counterpart (the additive cocycles of weight 2). Introduce modular symbols and explain [DV21, Cor. 1.10] which relates them to rigid meromorphic cocyles. Reference: [DV21, §1.1–1.3] [FLPSW23, Section 3].

Talk 5: Classification of rigid meromorphic cocycles of weight two (16.11., Theresa Häberle)
The goal of this talk is to classify additive rigid meromorphic cocycles of weight two. For that begin by briefly discussing rational cocylces and period functions as summarized in [DV21, 1.4]. Then discuss and give the main ideas for the proof of [DV21, Theorems 1.23 and 1.24]. Also explain [DV21, Corollary 1.27]. References: [DV21, §1.4–1.5].

Talk 6: The Schneider–Teitelbaum lift and the Dedekind–Rademacher cocycle (24.11., Judith Ludwig)
The goal of this talk is to describe the group $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$. For that first explain the so called Schneider–Teitelbaum lift ([DV22, Proposition 3.12]). Next introduce the Dedekind-Rademacher cocycle, [DPV23, §1]. Then discuss the structure of the space $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \otimes \mathbb{Q}$ as in [DV22, §3.7]. References: [DV22, §3], [DPV23, §1], [FLPSW23, §3.4.3].

Talk 7: RM values of rigid meromorphic cocycles (01.12., Gebhard Böckle)
The first goal of this talk is to discuss further structural results, in particular [DV21, Thm. 2.12]. The second aim is to explain how to evaluate rigid meromorphic cocycles at RM points of $\mathcal{H}_p$ and to discuss [DV21, Conj. 1] that builds the analogy between the RM values of rigid meromorphic cocyles and the singular moduli from the theory of complex multiplication. The last part should discuss more examples, namely the rigid meromorphic (theta) cocycles attached to RM points $\tau \in \mathcal{H}_p$. References: [DV21, §2, §3.1], [FLPSW23, §3].

Talk 8: Overview of p-adic modular forms (15.12., Alireza Shavali)
This talk should explain some background on overconvergent modular forms and p-adic families of (Hilbert) modular forms to prepare us for the next talks. Special focus should lie on families of Eisenstein series. A further goal is to explain Lemma 2.1 of [DPV21]. References: [DPV21], [FLPSW23, §5.1] and the references therein.

Talk 9: Families of Eisenstein series and the winding cocycle (12.01., Jakob Burgi)
The aim of this talk is twofold. First discuss the weight 2 overconvergent modular form $G_1'(\psi)$, defined as the first derivative of an incoherent Eisenstein family [DPV21, §2]. In a second part introduce the so called winding cocycle and expalin its properties as studied in [DPV21, §2.3–2.6]. References: [DPV21, §1-2], [FLPSW23, §3.3.2] and [DPV23, §2.1-2.2].

Talk 10: Diagonal restrictions of p-adic Eisenstein families (19.01., Lucas Gerth)
The goal of this talk is to prove [DPV21, Thm. B] and sketch the proof of [DPV21, Thm. C]. If time permits, also discuss the application to special values of twisted triple product p-adic L-functions. References: [DPV21, §2.7, §3–3.4], [FLPSW23, §5.2].

Talk 11: The RM values of the Dedekind–Rademacher cocylce (26.01., Oğuz Gezmiş)
The goal of this talk is to discuss the results of [DPV23], in particular Theorems C and B. References: [DPV23, §1], [FLPSW23, 5.3].

Talk 12: Rigid meromorphic cocycles for orthogonal groups I (02.02., Yingkun Li)
Describe the goal of the article [DGL23]. Explain how to attach to a non-degenerate quadratic space (V, q) over $\mathbb{Q}$ an achimedean and a p-adic symmetric space. Introduce the notion of a Kudla–Milson divisor. References: [DGL23, §1-2].

Talk 13: Rigid meromorphic cocycles for orthogonal groups II (09.02., Riccardo Zuffetti)
Define rigid meromorphic cocycles for orthogonal groups. If time permits sketch how to construct examples using a p-adic variant of Borcherd's singular theta lift. Define special points and explain how to evaluate rigid meromorphic cocycles at them. Formulate Conjecture 4.38 on the algebraicity of these values. References: [DGL23, §3-4].

References

[DD06] H. Darmon and S. Dasgupta, *Elliptic units for real quadratic fields*, Ann. Math. (2), 163(1): 301–346, 2006.

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[DV21] H. Darmon and J. Vonk, *Singular moduli for real quadratic fields: a rigid analytic approach*, Duke Math. J., 170(1): 23–93, 2021.

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[DPV21] H. Darmon, A. Pozzi, and J. Vonk, *Diagonal restrictions of p-adic Eisenstein families*, Math. Ann., 379(1-2): 503–548,2021.

[DPV23] H. Darmon, A. Pozzi, and J. Vonk, *The values of the Dedekind-Rademacher cocycle at real multiplication points*, J. Eur. Math. Soc., 2023.

[FLPSW23] P. Fust, J. Ludwig, A. Pozzi, M. Santos, and H. Wiersema, *Real quadratic singular moduli and p-adic families of modular forms*, preprint, available here.

[GZ85] B. H. Gross and D. B. Zagier, *On singular moduli*, J. Reine Angew. Math., 355:191–220, 1985.

[GZ86] B. H. Gross and D. B. Zagier, *Heegner points and derivative of L-series*, Invent. Math., 84:225–320, 1986.