Prof. Dr. G. Böckle, C.V. Sriram

Proseminar in Number Theory, SS2023

p-adic numbers

Tuesdays; 11 - 13 c.t. [1]

INTRODUCTION

For field K equipped with an *absolute value* $|.|$, one can form the *completion,* $\widehat{K}$ *of* K *w.r.t.* $|.|$, by considering the commmutative ring of all Cauchy sequences and reducing modulo the maximal ideal of null sequences. For instance, the field of rational numbers $\mathbb{Q}$ together with the usual absolute value $(x, y) \mapsto |x - y|$ gives rise to the field of real numbers $\mathbb{R}$.

There is a different *class* of absolute values on $\mathbb{Q}$, $|.|_p$ for each prime p, called the *p-adic absolute value*, which give rise to completions $\mathbb{Q}_p$ called the field of *p-adic numbers.*

In this preseminar, we aim at studying in detail, the construction of $\mathbb{Q}_p$ and some number theoretic applications(Hensel's lemma and local-global principle) in the first half(Talks 1-7). In the second half, we deal with $p-$adic analysis, beginning with sequences and series in $\mathbb{Q}_p$ and culminating in a talk on *p-adic Gamma function* Γ_p(Talks 8-13).

For more details about individual talks of the proseminar, please refer to the section on **TALKS** on next page.

PRACTICALITIES

1. Prerequisites. Students are expected to have a good understanding of the topics covered in Linear algebra 1 and Analysis 1.

2. Requirements of the participants.
(a) Delivery of seminar talk. The student is expected to deliver a 90-minute talk on their chosen topic. Definitions and results must be stated clearly, and where possible illustrated with concrete examples. Ideally the student will have worked through and understood all proofs; some of these proofs should be presented, though for reasons of time others may have to be omitted. Each presentation should contain at least one example.

(b) Preparation of handout. Please create a handout for the seminar participants to accompany your presentation. This document should contain the most important definitions and results from your presentation. You are also welcome to explain details of calculations and examples there for which there is no time in the lecture.

(c) Timelines for preparation of talk and handout.

(i) Start preparing your presentation in good time (approx. 4 weeks in advance). Come two weeks before your presentation for a preliminary discussion. The appointment for the preliminary discussion is then on the corresponding Tuesday at 9:30 a.m. Bring a draft of the handout to the preliminary meeting.
(ii) A template for handout is available on the seminar webpage.
(iii) Please email the completed handout to Mr. Sriram no later than Monday morning before your presentation, so that the materials can be made available to the other seminar participants in advance.

[1] Vorbesprechung on February 14, 2023 at 11:30 a.m.(s.t.)

(d) Language of talk and handout. The talks must be delivered in English. The handout should also be written in English.

TALKS

Part I: $p-$adic numbers

In the first part of this proseminar, we will construct the field of $p-$adic numbers, $\mathbb{Q}_p$ by setting up the necessary terminology. Talks 6 and 7 focus on some number theoretic applications of $\mathbb{Q}_p$. Main reference for this part is [Gou, Ch. 2 - Ch. 3].

1. Absolute values on fields

We start with the general definition of absolute value and introduce the $p-$adic absolute value. After showing that it is non-Archimedean we turn to properties of general absolute values and show different criteria for an absolute value to be non-Archimedean. Problems 26, 28 and 42 should be done for this talk. [2]

References: [Gou, §2.1-§2.2], supplementary [Rob, §2.1.3].

Lecture Date: April 18, 2022

Speaker:

2. Ultrametric topology and Valuation rings

In this lecture, we study in detail the ultrametric topology. First, the metric associated to an absolute value is defined. Next ultrametric spaces are defined and we will show that here all triangles are isosceles and all points within an open (or closed) ball are the centers of that ball. We also see that ultrametric spaces are totally disconnected.
We conclude this lecture with the definition of a valuation ring and the residue class field of a field w.r.t. non-Arschimedian valuation and determine these explicitly for the p-adic absolute value on $\mathbb{Q}$. For this we introduce the notions of a commutative ring and the maximal ideal and show that the quotient of a commutative ring with respect to a maximal ideal is a field. In general, it is useful at many points in this lecture to point out the many differences to the topology known from $\mathbb{R}$.

Reference: [Gou, §2.3 - §2.4]. For required definitions about rings, consult [Lan, §2.1-§2.2]

Lecture Date: April 25, 2023

Speaker:

3. Ostrowski's theorem and product formula

In this talk, we aim at the characterization of all absolute values of $\mathbb{Q}$, well known as the *Ostrowski's theorem*. First one studies the various notions of equivalence of absolute values. Then state and prove the Ostrowski's theorem(Problem 66 and 67 are useful for this part). Then we conclude the talk by proving the product formula, which says that the product of absolute values at all primes is 1.

References: [Gou, §3.1]. Also interesting to refer to [Kat, §1.2,§1.9]

Lecture Date: May 2, 2023

[2]The problems do not necessarily have to be explained in the lecture (the lecture time may be too short for that).

But the speaker is expected to have gone through these exercises as they mainly serve as a means for deeper understanding of the terminology, which has a direct impact on the lecture.

Speaker:

4. The field $\mathbb{Q}_p$

Here we construct the fundamental object of the proseminar, namely the field $\mathbb{Q}_p$. The construction is analogous to the construction of $\mathbb{R}$ from $\mathbb{Q}$, only now we use the $p-$adic absolute value. First we define the notion of a complete normed field. Then consider the ring of all Cauchy sequences and form the residue field reducing w.r.t. the maximal ideal of all null sequences, showing that it is the desired completion. We'll conclude by proving any two completions are uniquely isomorphic to each other. This helps us to focus on the properties of completion rather than the abstract construction presented before. Problem 79, 80 and 88 are instructive.

References: [Gou, §3.2].

Lecture Date: May 9, 2023

Speaker:

5. Properties of $\mathbb{Q}_p$

We start by defining and studying the *ring of $p-$adic integers* $\mathbb{Z}_p$, the closed unit ball in $\mathbb{Q}_p$. First prove some basic properties about $\mathbb{Z}_p$ and then show that it is compact topological space. Problems 96, 97 is helpful for recalling/understanding the topological notions involved here. Later we give two descriptions of $\mathbb{Z}_p$, as *coherent sequences* and as *$p-$adic expansions*. We end by discussing the units in $\mathbb{Z}_p$.

References: [Gou, §3.3]

Lecture Date: May 16, 2023

Speaker:

6. Hensel's lifting lemma

Historically, Kurt Hensel(1861-1941) was one of the first mathematicians to realize the importance of the field $\mathbb{Q}_p$ in arithmetic and formalize the notion[3]. In this talk we will study a result of his, of great consequence in number theory and elsewhere known as *Hensel's lifting lemma*. First we study the basic form of the lifting lemma, which says that generally one can lift "solutions over $\mathbb{Z}/p\mathbb{Z}$ to solutions over $\mathbb{Z}_p$" of polynomial equations. We apply this to get a description of the roots of unity in $\mathbb{Q}_p$. We conclude the talk by discussing a very general version of the lemma.

References: [Gou, §3.4]. One may also have a look at [Kat, §1.7].

Lecture Date: May 23, 2023

Speaker:

7. Local-Global principle

This is another crucial result of importance to number theorists, which roughly says "solutions exist over $\mathbb{Q}_p; \forall p \iff$ solution exists over $\mathbb{Q}$" for *general* diophantine equation. First formulate the slogan of the local global principle([Gou, Pg. 77]). Then present few counterexamples to the local-global principle. Problem 121 is instructive here. Then state the theorem for quadratic forms [Gou, Thm. 3.5.2]. In rest of the talk, we assume it to be true and study the equation $ax^2 + by^2 + cz^2$.

References: [Gou, §3.5]

[3] Just as an aside, in some works of E. Kummer(1810-1893), there is evidence of use of $p-$adic techniques

Lecture Date: May 30, 2023

Part II: p−adic analysis

In this part, we will be interested in exploring the similarity between $\mathbb{R}$ and $\mathbb{Q}_p$, by developing analytic tools, motivated by real numbers, to better understand the p−adic numbers. Lectures 8 to 10 mainly follow [Gou, Ch. 4]. For lecture's 11 and 12, [Kat, Ch. 4] is the main reference. Last lecture is based on [Kob, §IV.2].

8. Elementary analysis in $\mathbb{Q}_p$

We start the talk by stating and proving some results about sequence and series in $\mathbb{Q}_p$. Here one observes right away how the absolute value being non-Archimedean distinguishes it from the real analysis. Then we study power series over $\mathbb{Q}_p$. In particular we define radius of convergence of a power series and conclude the talk by showing that it does not change after changing the center [Gou, Prop. 4.2.3]. Some instructive problems: 138, 139.

References: [Gou, §4.1- §4.2]

Lecture Date: June 6, 2023

Speaker:

9. Strassman's theorem and the Logarithm function

We continue from the last talk, by studying a bound on the number of zeros of a power series known as *Strassman's theorem*[Gou, Thm. 4.2.4]. For instance, we see as a consequence that unlike sine and cosine function on $\mathbb{R}$, there are no periodic functions on $\mathbb{Q}_p$. Later we introduce the *logarithm function*[Gou, Defn. 4.3.2] and discuss its corresponding properties. We conclude by completing the structure theorem for the roots of unity in $\mathbb{Q}_p$, started in talk 6, by presenting problems 146, 147 and 148.

References:: [Gou, §4.2 - §4.3]. [Kat, §3.6] is also useful.

Lecture Date: June 13, 2023

Speaker:

10. Exponential function and $\mathbb{Z}_p^\times$

Analogous to logarithm function from last talk, one can also define the exponential function, although this requires more careful analysis of the radius of convergence. After discussing the preliminaries, we define the exponential function [Gou, Defn. 4.3.5]. Then we discuss how logarithm and exponential function behave with each other. Finally we give structure theorem for unit group $\mathbb{Z}_p^\times$ and briefly discuss the notion of a *Teichmüller lift*. Depending on time, one may discuss binomial expansions in $\mathbb{Q}_p$.

References: [Gou, §4.3].

Lecture Date: June 20, 2023

Speaker:

11. Continuity - Uniform continuity

We begin by defining continuous and uniformly continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$. Then we study *locally constant functions* and *step functions* by stating and proving an approximation result, of uniformly continuous functions by step functions. We then briefly discuss some obvious results about continuous functions(like sequential definition of continuity etc.) and cover some examples. Conclude the talk by proving an extension result for uniformly continuous functions, which is consequential for the next talk.

References: [Kat, §4.1 - §4.2]

lecture Date: June 27, 2023

Speaker:

12. The Interpolation method

The aim of this talk is to understand whether a function $f : \mathbb{N} \to \mathbb{Q}_p$ can be *interpolated* i.e. extends to a continuous function on $\mathbb{Z}_p$(if it does, it is unique), and if so construct an formula for the same. From last talk, we know this happens precisely when f is uniformly continuous. Here we give an explicit form of the extension known as *Mahler's theorem.* First define what it means for a function to be interpolated and give equivalent conditions. Exhibit non-constant Cauchy sequences as counterexamples to interpolation. Then develop *interpolation coefficients* and *interpolation series* of $f : \mathbb{N} \to \mathbb{Q}_p$ and prove *Mahler's theorem.* Conclude by proving *Weierstrass preparation theorem* and some properties of the exponential function.
References: [Kat, §4.6]

lecture Date: July 4, 2023

Speaker:

13. $p-$adic Gamma function

In the final talk, we would like to study another application of the interpolation series(from talk 12), the *$p-$adic Gamma function.* We begin the talk from the classical Gamma function Γ, its definition(as an *integral representation*), the functional equation($\Gamma(x + 1) = x\Gamma(x)$) and also discuss the fact that it interpolates the factorial numbers $n!$([Rud, Defn. 8.17, Thm. 8.18]). Then state the formula from [Coh, Prop. 9.6.17](without proof) and use it as a motivation to define the $p-$adic Gamma function in [Coh, Defn. 11.6.5]. Prior to defining, one must show that the sequence is uniformly continuous to apply [Kat, Prop. 4.39], which is done in [Coh, Prop. 11.6.4]. Discuss some properties of Γ_p from [Coh, 11.6.7] and [Coh, 11.6.8](depending on time). Finally conclude by determining the interpolation series of Γ_p from [Coh, Prop. 11.6.15(1)].
References: [Rud, Ch. 8], [Gou, §4.6], [Coh, Ch. 9, Ch. 11].

lecture Date: July 11, 2023

Speaker:

References

[Coh] Cohen, Henri. (2007). Number theory. Volume II: Analytic and modern tools. 10.1007/978-0-387-49894-2.

[For] Forster, O. (2008). Analysis 1. Differential- und Integralrechnung einer Veränderlichen. 978-3-8348-9464-9

[Gou] Gouvea, F.: p-adic Numbers: An Introduction. Springer University text, ISBN 978-3-662-22278-2 (eBook) .

[Kat] Katok, S.: p-adic Analysis Compared with Real. AMS Student Mathematical Library 37 (2007).

[Kob] Koblitz, N.: p-adic Numbers, p-adic Analysis and Zeta-Functions. Springer Graduate Texts in Mathematics 58 (1984).

[Lan] Lang, S.: Algebra. Springer Graduate Texts in Mathematics 211 (2005).

[Rob] Robert, AM: A Course in p-adic Analysis. Springer Graduate Texts in Mathematics 198 (2000).

[Rud] Rudin, Walter: Principles of Mathematical Analysis.