Hauptseminar *Arithmetik von Zahl- und Funktionenkörpern* WS 2017/18

Big image theorems for Hida families

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Introduction

Let f be a classical modular form of level N and some weight. Suppose that f is cuspidal and a Hecke eigenform. For every prime number p that does not divide the level of f, denote by $a_p(f)$ the Hecke eigenvalue of f at p. Let $E = E_f$ be the coefficient field of f, i.e., the extension of $\mathbb{Q}$ that is generated by the Hecke eigenvalues $a_p(f)$ for all p not dividing N. It is known that E is a number field. Choose an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. For every prime p, choose an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ into the algebraic closure $\overline{\mathbb{Q}}_p$ of the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ at p. This defines an embedding $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. Choose an automorphism $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}$ that maps to the geometric Frobenius automorphism in $G_{\mathbb{F}_p}$ under the canonical homomorphism $G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}$.

The arithmetic theory of the modular curve allows one to attach to f a compatible system of continuous Galois representations

$$\rho_\lambda = \rho_{f,\lambda} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$$

for every finite place λ of E, and where E_λ denotes the completion of E at λ. What makes the system compatible and at the same time relates it to f is the Eichler-Shimura relation

$$\mathrm{Trace}\,\rho_\lambda(\mathrm{Frob}_p) = a_p(f) \qquad \text{for all primes } p \text{ not dividing } N\ell_\lambda.$$

where ℓ_λ is the prime number under λ, and where one embeds $a_p(f)$ via the completion homomorphism $E \hookrightarrow E_\lambda$ into E_λ. By choosing a suitable conjugate, we may and will assume that ρ_λ takes its image in $\mathrm{GL}_2(\mathcal{O}_\lambda)$, where $\mathcal{O}_\lambda$ is the valuation ring of E_λ.

If f is the form Δ of weight 12 and level 1, so that $E = \mathbb{Q}$, then the non-surjectivity of $\rho_\ell \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ is related to number-theoretic properties (congruences) of the Ramanujan τ-function. If f is of weight 2 and related to an elliptic curve $A/\mathbb{Q}$, then Serre shows that $\rho_{f,\ell}$ is surjective for almost all ℓ and has open image in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all ℓ, provided that A does not have complex multiplication (CM). If A has CM, the image can also be described up to finite error. In the CM case the curve A has extra endomorphisms over a finite extension K of $\mathbb{Q}$. The Galois action of G_K on the ℓ-adic Tate-module of A has to commute with these endomorphisms. It is precisely this Galois action that is described by $\rho_{f,\lambda}$, and it turns out that the centralizer of the extra endomorphism is a non-split torus T in $\mathrm{GL}_{2.\mathbb{Q}}$. Hence the $\rho_\ell|_{G_K}$ take their image in $T(\mathbb{Z}_\ell)$, and with at most finitely many exceptional ℓ the image is equal to $T(\mathbb{Z}_\ell)$.

For general f one can define a Mumford-Tate group M_f over $\mathbb{Q}$ which is a closed subgroup of the Weil restriction $\mathrm{Res}_{E/\mathbb{Q}}\,\mathrm{GL}_{2,E}$, and it is proven that $\prod_{\lambda|\ell} \rho_\lambda|_{G_K}$ has open image in

$M_f(\mathbb{Q}_\ell)$ for a suitable number field K independent of ℓ. The group M_f is vaguely speaking made up of symmetries of f. For instance, if f has so-called inner twists, then the image of $\prod_{\lambda|\ell} \rho_\lambda$ is significantly smaller than $\prod_{\lambda|\ell} \mathrm{GL}_2(E_\lambda)$, as shown by Momose and Ribet. In general one has the following guiding principle, e.g. [8, Ch. 1]:

Heuristic: The image of a Galois representation that arises from a geometric object (such as an elliptic curve, modular form, or motive) should be as large as possible, subject to the symmetries of the geometric object from which it arises.

The aim of the seminar is to learn about recent results of the above principle first by H. Hida and then by J. Lang not only for a single modular form but for all forms in what is called a Hida family. In this case one fixes a prime ℓ. The representation takes values in $\mathrm{GL}_2(R)$ where R is local Noetherian and a finite flat extension of $\mathbb{Z}_\ell[[t]]$. The image need not be open in R. But as shown by Lang (and for $R = \mathbb{Z}_\ell[[t]]$ by Hida) the image contains $K_{R,I} := \ker(\mathrm{SL}_2(R) \to \mathrm{SL}_2(R/I))$ for I a non-zero ideal of R. If the ideal I is not open in R, then Hida (morally) shows that this comes from specializations of the Hida family at which the image is not open in $\mathrm{GL}_2(E_\lambda)$. In certain cases he then goes on to give relations to p-adic L-functions.

The seminar begins with an introduction to Hida families. Then it presents Theorem I of Hida from [6], the containment of $K_{\mathbb{Z}[[t]],I}$ in the image of the representation of a Hida family in $\mathrm{GL}_2(\mathbb{Z}_\ell[[t]])$, assuming that the Hida family does not have CM. Next we present the generalization indicated above if this, i.e., the main result of Lang from [8]. The extra endomorphisms come from inner twists of the Hida family f. The seminar ends by indicating the relations between properties of p-adic ℓ-functions and the size of I in $K_{R,I}$.

1 Talks

Talk 1 (Overview + Organization)**.** This talk will give an overview of the history and the development of big image results. The speaker should start with the heuristics of [8, Section 1] (also include Example 1) and continue with the classical setup (non-CM cuspidal eigenforms) as studied by Ribet and Momose. After a brief definition of Hida families, state the main results of [6]. Then discuss the extensions of [8] and [2]. Conclude the talk with a brief sketch of the work in progress by Lang and with applications of big image theorems for Euler systems and the inverse Galois problem (see the end of Section 1 of [8]).
Duration: 60-70 minutes. After the talk we will collect preferences and distribute the remaining talks accordingly.
Date: October 19, 2017 **Speaker: David**

Talk 2 (First background talk: Classical modular forms)**.** In the first part, you will remind the audience about the standard definitions and facts concerning classical modular forms, e.g. from [4] or [9]. Keywords that should appear: Definition of a modular form, cuspidality, q-expansion, Hecke operator/ algebra/ eigenform, newform, the duality between modular forms and Hecke algebras, conjugate self-twist and CM (Definition 3.1.1 of [8]). Briefly outline the association of Galois representations to eigenforms, e.g. using Section 3.1 of [5] (or, more specifically, Theorem 2.3.1 of [8]).

In the second part, we will examine the historically first example of a big image result. For this, reproduce as much material as possible from §3 of [11], in particular sketch the proof of Theorem 10. Finish your talk with some examples from the end of §3.
Date: October 26, 2017 **Speaker: Q. Gazda**

Talk 3 (Second background talk: p-adic modular forms and Hida families)**.** In this talk, the speaker will cover all of Chapter 2 of [8], excluding what has been explained in the previous talk. In particular, cover p-adic modular forms, the p-adic Hecke algebra and its ordinary part, the structure of the Hecke algebra as a Λ-module (Theorem 2.2.2). Next, give the definition of a Hida-family (Definition 2.1.1) and carefully explain the connections to $\mathbb{I}$-adic cusp forms (Chapter 2.2.3). (You can also illustrate this with material from Sections 4.3-4.5 of [10].) Conclude with the theory of newforms in Hida families, in particular (and if time permits) outline the proofs of Theorem 2.3.2 and Theorem 2.3.3. (As these objects will be used not only in the context of [8] but also for [6], you should connect to the speaker of talk 4 to (at least) align notation.)

Date: November 2, 2017 **Speaker: A. Troya**

Talk 4 (Hida's Theorem I, part 1)**.** Introduce the audience into the setting of the article [6] by carefully explaining the introduction up to p. 605. In particular, state conditions (F), (R), (s), (u) and (v) and the connections between them and make clear what we assume for the remainder. Finish your exposition on the Introduction-chapter by stating Theorem I. Next, cover all necessary material about Pink's theory of Lie algebras for p-finite subgroups from Section 1. You will not have time for many proofs, but at least the statements of Theorem 1.1, Lemma 1.3 and Lemma 1.4 should appear. As this part is heavy on notation, you might might want to prepare a handout for the audience. (The speaker of talk 7 will also make use of this theory, so you should coordinate your preparation. On the other hand, talks 4 and 7 might be a good choice for one and the same speaker.) (For this and all other talks on [6], the slides [7] can serve as a first introduction into the paper.)

Date: November 9, 2017 **Speaker: Özge**

Talk 5 (Hida's Theorem I, part 2)**.** This talk covers the fundamental fullness-result of Section 2 of [6]. Lemma 2.9 and Theorem 2.12 are essential for the remainder (also concerning Theorem II), so you should follow a top-down approach: Make sure you state these two results including all necessary notation, and spend the rest of the time on a sketch of their proofs.

Date: November 16, 2017 **Speaker: ?**

Talk 6 (Hida's Theorem I, part 3)**.** This talk should cover the main parts of Section 3 of [6]: Prove Lemma 3.1 and Lemma 3.2 and deduce (the main part of) Theorem I. Give a sketch on how the remaining part (uniqueness of (L)) is shown. Be sure to introduce the global level $L(\mathbb{I})$ and cover Lemma 3.5, as this will be used in the proof of Theorem II later on.

Date: November 23, 2017 **Speaker: ?**

Talk 7 (Lang's Thesis, part 1)**.** The purpose of the next three talks is the proof of the main result (Theorem 3.1.4) of [8]. For this, first give the statement of Theorem 3.1.4 and then outline the strategy of the proof as on p. 22 (but see also the summary on p. 57). Proceed to Proposition 3.3.2 and cover as much of the proof (i.e. of Section 3.3) as possible. (Also oberve the connections to talk 4!) Another key argument (used in the next talk) is Theorem 3.2.1, with which you should conclude your talk. You will not have enough time to give the proof (this is the content of Section 3.2), but give a summary of the methods used (see p. 24). Also give a remark towards Section 4, where Lang gives an alternative proof, avoiding deformation theory.

Date: November 30, 2017 **Speaker: ?**

Talk 8 (Lang's Thesis, part 2)**.** This aim of this talk is to give the statement of Proposition 3.4.1 in [8] and as much of its proof as possible. After the statement of Proposition 3.4.1 and a reminder on its application in the proof of Theorem 3.1.4, explain Lemma 3.4.2 and the isomorphism statement Proposition 3.4.3 (this is where Goursat's Lemma is used!). Proceed to obstruction theory as given in the main body of Section 3.4, yielding Proposition 3.4.4, Lemma 3.4.5 and Lemma 3.4.6 (for time constraints you will probably only be able to give rough sketches of the arguments used). Recall Theorem 3.2.1 from the last talk and put together the proof of Proposition 3.4.1 (see p. 53).
Date: December 7, 2017 **Speaker: ?**

Talk 9 (Lang's Thesis, part 3)**.** The topic of this talk is how the material of the last two sections is put together to the proof of Theorem 3.1.4. Both for motivation and for its use in the proof of Theorem 3.1.4, state the result of Ribet-Momose on images of classical modular forms, cf. Theorem 3.5.1 in [8]. Proceed with the crucial Theorem 3.3.1 (where, again, for time constraints you will only be able to give a rough sketch of the 7-page-long proof given in Section 3.6). Proceed to Proposition 3.5.2 and Corollary 3.5.3 (and its proofs, if time permits) and conclude the talk by carefully putting together the proof of Theorem 3.1.4 (as outlined on p. 57).
Date: December 14, 2017 **Speaker: ?**

Talk 10 (Hida's Theorem II, part 1)**.** Here the speaker will prove part (1) of Theorem II in [6], which is basically [6, Theorem 8.2]. You should begin with a general motivation of the problem that Theorem II wants to solve: To relate the global level $L(\mathbb{I})$ (defined in talk 6) with p-adic L-functions (cf. the text between Lemma 8.1 and Theorem 8.2). The proof of Theorem 8.2 can then be done in a straightforward way following Section 8. First recall conditions (R), (s) and (v) and Lemma 2.9 and prove Lemma 8.1. Then carefully go through the proof of Theorem 8.2. Finish your talk with Remark 8.3.
Date: December 21, 2017 **Speaker: ?**

Talk 11 (Hida's Theorem II, part 2)**.** This talk serves as an introduction to the L-functions appearing in the statements of Theorem II. For Katz p-adic L-functions use [3] and for Kubota-Leopoldt p-adic L-functions use material from [1]. Although [6] is not a direct reference for this talk, you should coordinate the selection of topics you cover with the speakers of the following two sessions.
Date: January 11, 2018 **Speaker: Michael Fütterer**

Talk 12 (Hida's Theorem II, part 3)**.** Here, the speaker will present Theorem II, (3b) (which is part (a) of Theorem 8.5 in [6]). As the proof is long and involves many results from earlier sections of the article (Lemma 5.5-Proposition 5.7, Theorem 7.1 and in particular Theorem 7.2), the best would be to explain the (1,5-page long) summary between the statement of Theorem 8.5 and its proof, referring to and explaining earlier results as they are used. If time permits, conclude your talk with the statement of Theorem 8.6 and a remark on the analogy of its proof with the proof of Theorem 8.6.
Date: January 18, 2018 **Speaker: Michael + Peter**

Talk 13 (Hida's Theorem II, part 4)**.** Start with the relatively short proof of part (2) of Theorem II in [6] (Section 8, Theorem 8.7). Proceed with a summary of the proof of Theorem 8.8 which will give part (4) of Theorem II. If time permits, you can present the additional (conjectural) cases not included in Theorem II, i.e. Conjecture 9.1 and explain the reasoning supporting this conjecture.
Date: January 25, 2018 **Speaker: Peter**

Talk 14 (RESERVE DATE)**. Date: February 1, 2018** **Speaker: ?**

Talk 15 (RESERVE DATE)**. Date: February 8, 2018** **Speaker: ?**

References

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[2] A. Conti, A. Iovita and J. Tilouine (2016); *Big image of Galois representations associated with finite slope p-adic families of modular forms*, in Elliptic Curves, Modular Forms and Iwasawa Theory: In Honour of John H. Coates' 70th Birthday, Cambridge, UK, March 2015, edited by D. Loeffler and S. L. Zerbes, Springer Proceedings in Mathematics & Statistics, Vol. 188, Springer, 2016 (preprint version available at `https://sites.google.com/site/andreacontimath/research`)

[3] E. de Shalit (1987); *Iwasawa theory of elliptic curves with complex multiplication*, Perspectives in Mathematics, 3. Academic Press, Inc., Boston, MA.

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[5] H. Hida (03/2017); *Modular Forms and their Galois Representations*, available at `http://www.math.ucla.edu/~hida/205b.1.17w/Lec17W.pdf`

[6] H. Hida (2015); *Big Galois Representations and p-adic L-functions*, Compositio Math. 151 (2015), 603-664 (preprint version available at `http://www.math.ucla.edu/~hida/`)

[7] H. Hida (2011); *Big Galois Representations and p-adic L-functions First lecture*, lecture notes available at `http://www.math.ucla.edu/~hida/ParisLie_sld.pdf`

[8] J. Lang (2016); *Images of Galois representations associated to p-adic families of modular forms*, PhD thesis at UCLA, available at `http://www.math.univ-paris13.fr/~lang/`

[9] W. Li (2006); *Lectures on Modular Forms and Galois Representations*, lecture notes available at `http://math.cts.nthu.edu.tw/Mathematics/lecnotes/2006`

[10] B. Mazur (2012); *A brief introduction to the work of Haruzo Hida*, available at `http://www.math.harvard.edu/~mazur/papers/Hida.August11.pdf`

[11] J.-P. Serre (1971); *Congruences et formes modulaires*, available at `http://www.numdam.org/article/SB_1971-1972__14__319_0.pdf`