On the density of modular points in universal deformation spaces

by

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Abstract

Based on comparison theorems for Hecke algebras and universal deformation rings with strong restrictions at the critical prime l, as provided by the results of Wiles, Taylor, Diamond, et al., we prove under rather general conditions that the corresponding universal deformation spaces with no restrictions at l can be identified with certain Hecke algebras of l-adic modular forms as conjectured by Gouvêa, thus generalizing previous work of Gouvêa and Mazur.

Along the way, we show that the universal deformation spaces we consider are complete intersections, flat over $\mathbb{Z}_l$ of relative dimension three, in which the modular points form a Zariski dense subset. Furthermore the fibers above $\mathbb{Q}_l$ of these spaces are generically smooth.

1 Introduction

When first defining universal deformation spaces for deformations of Galois representations in the seminal paper [22] of Mazur, their complete arithmetic meaning may have been rather obscure. Motivated by the work of Hida, [19], Mazur was led to expect the density of ordinary modular representations in the universal ordinary deformation space. Over time however the understanding improved and at least for odd irreducible two dimensional residual modular representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ concrete conjectures were made in [14] and [15] by Gouvêa.

In a coarse form, these conjectures predict that the universal deformation spaces are isomorphic to Hecke algebras of l-adic modular forms. The precise form will be given below in Conjecture 2.9. Loosely speaking, we will call the above deformation spaces and Hecke algebras the big ones. As a consequence of these conjectures, [18], §3, one can reconstruct universal deformation spaces entirely from the set of modular forms of a given mod l reduction type and a bound on the prime-to-l conductor.

The first progress on these conjectures was made in [18] by Gouvêa and Mazur, who proved them under certain smoothness assumptions on the spaces involved. An essential ingredient was the work of Coleman, [8] on families of overconvergent modular forms. Based on an observation in [1], we can vastly extend the range in which the above conjectures are true.

The new ingredient compared to [18] is the use of the isomorphism theorems in [29], [28] and [13] between universal deformation rings and Hecke algebras with strong supplementary conditions at the place l. These rings will be called the small rings. Under some conditions on the residual representation at l, we will show that there is an isomorphism of the big rings whenever there is an isomorphism of the small ones, and this is where the results of [1] come to bear.

The proof proceeds in two stages. First, we use the isomorphism results between the small rings and the way the small universal rings are obtained from the big ones, as explained in [1], to derive some consequences on the geometry of the big universal deformation rings: They are complete intersections and each component of their associated space contains a smooth point which is modular (of weight 2 and finite slope) and not contained in any other component. Therefore the big deformation spaces contain a dense open subspace of characteristic zero which is formally smooth over $\mathbb{Q}_l$. We simply say that the big spaces are *generically smooth*. This can be seen as partial confirmation of some conjectures of Boston, cf. Remark 3.6 and [3].

The second step will be a reexamination of the density proof in [18]. The proof in loc.cit., with minor modifications, shows in fact the following. Suppose we have a smooth modular point of finite slope on a component of a big universal space, and a neighborhood of it which meets no other component. Then the modular points are Zariski dense in this component.

Combining the preceding two steps, we find that modular points are Zariski dense in big universal deformation spaces under the assumption that the residual representation is modular, absolutely irreducible and satisfies certain conditions at l. As the big spaces are generically smooth, it easily follows that the big universal rings are isomorphic to their corresponding Hecke algebra.

It should certainly be investigated if or under what circumstances the ideas of this article generalize to situations where either $\bar{\varrho}$ is reducible, or $\bar{\varrho}$ is an irreducible representation of the absolute Galois group of a totally real field and attached to a Hilbert modular form.

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2 Deformation rings and Hecke algebras

In this section, we fix the type of residual representation we want work with. This is followed by a brief discussion on ordinariness. We then introduce various deformation rings and corresponding Hecke algebras. In the end, we state Gouvéa's conjecture.

Let $l \in \mathbb{N}$ be an odd prime and $\mathbb{F}_l$ the field of l elements. For an arbitrary field K, by $\bar{K}$ we denote its separable closure. We abbreviate $\mathrm{Gal}(\bar{K}/K)$ by G_K. Let $S \supset \{l, \infty\}$ be a finite set of places of $\mathbb{Q}$, and denote by $G_{\mathbb{Q},S}$ the Galois group of the maximal subextension of $\bar{\mathbb{Q}}$ which is unramfied outside S. For every rational prime p we choose a place of $\bar{\mathbb{Q}}$ above it. This provides us with an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ and a corresponding map $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$. The image of the latter map is denote by D_p, and its inertia subgroup by I_p. Let $\chi\colon G_{\mathbb{Q}} \to \mathbb{Z}_l^*$ be the cyclotomic character, $\bar{\chi}$ its reduction mod l, and ω the Teichmüller lift of $\bar{\chi}$. By $\chi_p, \bar{\chi}_p, \omega_p$ we denote the respective restrictions to D_p.

The residual representation

We start by specifying a residual representation. Fix an odd continuous Galois representation $\bar{\varrho}' : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_l)$. (We will shortly introduce $\bar{\varrho}$ which is the simplest twist of $\bar{\varrho}'$ by a character; so the use of $\bar{\varrho}'$ is only temporarily.) By continuity, the fixed field of the kernel of $\bar{\varrho}'$ is a finite extension of $\mathbb{Q}$, and thus $\bar{\varrho}'$ is ramified at only finitely many places. By $\mathrm{Ram}(\bar{\varrho}')$ we denote the set of such. The underlying two-dimensional representation space is denoted by $\bar{W}'$. The restriction of $\bar{\varrho}'$ to D_p for any rational prime p is denoted by $\bar{\varrho}'_p$.

Assumption 2.1 Throughout, we require that $\bar{\varrho}'$ satisfy the following conditions:

(i) $\bar{\varrho}'$ restricted to $G_{\mathbb{Q}\left(\sqrt{(-1)^{(l-1)/2}l}\right)}$ is irreducible.

(ii) $\bar{\varrho}'$ is modular in the sense that it arises from a modular form, cf. [27].

Furthermore, we require $\bar{\varrho}'_l$ to satisfy either of the following two conditions.

(iii) $\bar{\varrho}'_l$ is reducible, but the image of its semisimiplification is not in the center of $\mathrm{GL}_2(\bar{\mathbb{F}}_l)$.

(iv) There exists a character ν_l of $G_{\mathbb{Q}_l}$ such that $\bar{\varrho}'_l \otimes \nu_l$ is irreducible and flat in the sense of [29], p. 456.

If $\bar{\varrho}'_l$ is as in (iii), we can write $\bar{\varrho}'_l = \begin{pmatrix} \bar{\chi}_{1,l} & * \\ 0 & \bar{\chi}_{2,l} \end{pmatrix}$ with respect to a suitable basis. If $l = 3$, or if $\bar{\varrho}'_l$ is not semi-simple, we fix any such basis. If $\bar{\varrho}'_l$ is semi-simple and $l \neq 3$, we choose a basis such that $\bar{\chi}_{1,l}^{-1}\bar{\chi}_{2,l} \neq \bar{\chi}_l$. This basis will be fixed throughout the article and based on it, we define the adjoint character of $\bar{\varrho}'_l$ to be $\bar{\mu}'_l := \bar{\chi}_{1,l}^{-1}\bar{\chi}_{2,l}$. So really, the adjoint character may depend on our choice of basis and not just on $\bar{\varrho}'$. Our special choice of basis in the semi-simple case will is needed to have Theorem 3.1 at our disposal.

For the deformation theory to be developed below, we want to work with a residual representation which takes its values in $\mathrm{GL}_2(k)$ for a finite extension k of $\mathbb{F}_l$. Since $\bar{\varrho}'$ and ν_l have finite image we can clearly assume this. By possibly passing to the unique quadratic extension of k, we will furthermore assume that any semisimple element in the image of $\bar{\varrho}'$ can be diagonalized over k.

By $W(k)$ we denote the ring of Witt vectors over k, by K some finite extension of the fraction field of $W(k)$ and by $\mathcal{O}$ its ring of integers. By $|\,.\,|$ we denote the norm on K such that $|l| = 1/l$, and by $v(\,.\,)$ the corresponding valuation. We extend both of them to $\bar{\mathbb{Q}}_l$ via a fixed embedding $K \hookrightarrow \bar{\mathbb{Q}}_l$.

Ordinariness

An important condition on two-dimensional Galois representation is ordinariness. There are two reasons why we include a brief discussion on this. First, the conjectural isomorphism between big universal rings and big Hecke algebras made in

[15] involves the notion of p-ordinariness for primes p different from l. Second, the definition of ordinariness is not uniform in our references.

We let A be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k and fix a continuous Galois representation $\varrho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$. We write W for the underlying free A-module of rank two. For a subgroup H of $G_{\mathbb{Q}}$, we denote by W_H the covariants and by W^H the invariants of W with respect to H.

Definition 2.2 *For p any rational prime, we say that ϱ is* p-ordinary *if W_{I_p} is free of rank one over A. It is* p-co-ordinary *if W/W^{I_p} is free of rank one over A and if the same holds for the reduction of W modulo $\mathfrak{m}$.*

Suppose that $\mathrm{Im}\,\varrho_p$ is not in the center of $\mathrm{GL}_2(A)$. We say that ϱ is weakly p-ordinary *(respectively* weakly p-co-ordinary*) if with respect to a suitable A-basis of W one has*

$$\varrho_{|D_p} = \begin{pmatrix} \chi_{1,p} & * \\ 0 & \chi_{2,p} \end{pmatrix}$$

where $\chi_{i,p} : D_p \to A$ are continuous characters and $\chi_{2,p}$ (respectively $\chi_{1,p}$) is unramified.

Remark 2.3 Clearly, if ϱ is p-ordinary (p-co-ordinary), it is weakly p-ordinary (weakly p-co-ordinary). What we call weakly l-ordinary is called ordinary in [29], p. 456.

Compared to [23] and [15] the notions of ordinariness and co-ordinariness are interchanged. To be precise, in those two references ϱ is called p-ordinary if and only if W/W^{I_p} is a free A module of rank one. In all cases where we quote from [15], it is assumed that $\varrho(\mathrm{mod}\ \mathfrak{m})$ has the respective property too. No confusion should arise.

The following lemma clarifies the relation between the above two notions.

Lemma 2.4 *(a) A representation ϱ is p-ordinary if and only if $\varrho(\det\varrho)^{-1}$ is p-co-ordinary. The same holds for the corresponding weak notions.*

(b) Suppose p is a prime different from l, then ϱ is p-ordinary if and only if it is p-co-ordinary.

Proof: Part (a) is left as an exercise to the reader. We now prove part (b) and assume that ϱ is p-ordinary. Thus with respect to a suitable basis of W, one can write

$$\varrho_{|I_p} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

with continuous functions $a, b\colon I_p \to A$. From the continuity of ϱ and the structure of $\mathrm{GL}_2(A)$, it follows immediately that $\varrho(I_p)$ is the extension of a finite cyclic group L' of order prime-to-l by a pro-l group L.

It is also well-known that I_p is the extension of a pro-cyclic group of order prime-to-p by a pro-p group. Therefore, $\varrho(I_p)$, too, is the extension of a pro-cyclic group P' of order prime-to-p by a pro-p group P. As p is different from l, the induced map $\varrho(I_p) \to P' \times L'$ has trivial kernel, which shows that $\varrho(I_p)$ is abelian. This implies that L is a quotient of P', and hence of $\mathbb{Z}_l$. We obtain that $\varrho(I_p) \cong L \times L'$ is topologically generated by a single element, say $g_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & 1 \end{pmatrix}$.

As W_{I_p} is free of rank one over A the subspace $(g_0 - \mathrm{id})W$ of W contains the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus there exist $v_1, v_2 \in A$ such that $(a_0 - 1)v_1 + b_0 v_2 = 1$.

Performing a base change with the matrix $h := \begin{pmatrix} -v_1 & b_0 \\ -v_2 & 1-a_0 \end{pmatrix}$ yields $h^{-1}g_0h = \begin{pmatrix} a_0 & 0 \\ -v_2 & 1 \end{pmatrix}$. A short calculation shows that W^{I_p} is free over A of rank one and a direct summand of W, and the same for $W \pmod{\mathfrak{m}}$. Hence ϱ is p-co-ordinary.

We now turn to the converse. Arguing as above, the p-co-ordinariness of ϱ implies that $\varrho(I_p)$ is topologically generated by one generator, which we call g_1. We chose a basis such that $g_1 = \begin{pmatrix} a_1 & 0 \\ c_1 & 1 \end{pmatrix}$. Our hypothesis that $\varrho \pmod{\mathfrak{m}}$ is p-co-ordinary, implies that one of $a_1 - 1$ and c_1 is a unit in A. Therefore there exist w_1, w_2 in A such that $w_1(a_1 - 1) + w_2c_1 = 1$. This allows one to revert the above argument and to show that ϱ is p-ordinary. ∎

As indicated above, we may have to replace $\bar{\varrho}'$ by a twist which is simpler in certain ways. To do so, we first define local characters ψ'_p at all primes p.

(a) If $p \neq l$ and $\bar{\varrho}'_p$ is irreducible or unramified, we let ψ'_p be trivial.

(b) If $\bar{\varrho}'_p$ is reducible and ramified, we choose ψ'_p such that $(\bar{W}' \otimes \psi'_p)^{I_p} \neq 0$.

(c) If $p = l$ and $\bar{\varrho}'_l$ is irreducible, we set $\psi'_l := \nu_l^{-1}$ — recall that ν_l is a character of $G_{\mathbb{Q}_l}$ such that $\varrho'_l \otimes \nu_l$ is flat.

Now we define the character $\psi\colon G_{\mathbb{Q}} \to k^*$ to be the unique character such that for all primes p, the local characters ψ_p and ψ'_p agree when restricted to I_p. We let $\bar{\varrho} := \bar{\varrho}' \otimes \psi$. In particular this means that $\bar{\varrho}$ is p-ordinary or unramified at all primes p where $\bar{\varrho}'_p$ is ramified and reducible. Note that Assumption 2.1 continues to hold for $\bar{\varrho}$. Also, if $\bar{\varrho}'_l$ is reducible, then $\bar{\varrho}_l$ is weakly l-ordinary and has adjoint character $\bar{\mu}_l = \bar{\mu}'_l$.

Universal deformation rings

We let Σ be a finite set of places of $\mathbb{Q}$ which contains l, ∞ and all the primes p at which $\bar{\varrho}_p$ is ramified but not p-ordinary. We define $S := \Sigma \cup \mathrm{Ram}(\bar{\varrho})$. Thus at all $p \in S\backslash\Sigma$ the representation $\bar{\varrho}$ is p-ordinary. Depending on the set Σ, we define various deformation rings for $\bar{\varrho}$ and $\mathcal{O}$ as above. For generalities on universal deformation rings and their properties, we refer to [22], or [7]. We also choose Σ' for $\bar{\varrho}'$, and define S' analogously. At this point there is no relation between Σ' and Σ.

By $\mathcal{R}_{\mathcal{O},\Sigma}$ we denote the universal deformation ring for deformations of $\bar{\varrho}$ to complete noetherian local $\mathcal{O}$-algebras, unramified outside S which are p-ordinary at all primes p of $S\backslash\Sigma$. The definition of being p-ordinary is clearly invariant under conjugation and hence a well-defined condition on deformations of Galois representations, cf. [22]. Using $\bar{\varrho}', S', \Sigma'$ instead, we define the ring $\mathcal{R}'_{\mathcal{O},\Sigma'}$.

The quotient of $\mathcal{R}_{\mathcal{O},\Sigma}$ which classifies deformations $[\varrho]$ of $\bar{\varrho}$ such that $\det\varrho$ is of finite order prime to l at any prime p different from l is denoted by $\mathcal{R}^l_{\mathcal{O},\Sigma}$. (Note that $\det\varrho$ is independent of the chosen representative of $[\varrho]$.) Finally, by $\mathcal{R}^s_{\mathcal{O},\Sigma}$ we denote the quotient of $\mathcal{R}^l_{\mathcal{O},\Sigma}$ which parameterizes deformations that satisfy the following conditions at l, cf. [29], pp. 456–457:

(a) The order of $(\det\varrho_l)\chi_l^{-1}$ is finite and prime to l.

(b) In case (iii), ϱ_l is weakly l-ordinary, in case (iv), ϱ_l is a flat deformation.

We set $X_\Sigma := \mathrm{Spec}(\mathcal{R}_{\mathcal{O},\Sigma})$, $X'_{\Sigma'} := \mathrm{Spec}(\mathcal{R}'_{\mathcal{O},\Sigma'})$ and $X^l_\Sigma := \mathrm{Spec}(\mathcal{R}^l_{\mathcal{O},\Sigma})$ to denote the corresponding spaces.

Remark 2.5 There are several comments in order to motivate the definitions of the above deformation rings.

(a) The ring $\mathcal{R}_{\mathcal{O},\Sigma}$ (respectively $\mathcal{R}'_{\mathcal{O},\Sigma'}$) we want to identify with an l-adic Hecke algebra of l-adic modular forms as constructed in [14]. In loc.cit. this ring is denoted by $\mathbf{R}(\bar\varrho, S, S\backslash\Sigma)$.

(b) The ring $\mathcal{R}^l_{\mathcal{O},\Sigma}$ is linked to $\mathcal{R}^s_{\mathcal{O},\Sigma}$ by imposing conditions solely at the prime l, see Theorem 4.1 below.

(c) The ring $\mathcal{R}^s_{\mathcal{O},\Sigma}$ is isomorphic to the ring $R_{\mathcal{D}}$ of [29], p. 458, where in case (iii) we have $\mathcal{D} = (\mathrm{Se}, S, \mathcal{O}, \Sigma)$, and in case (iv) we have $\mathcal{D} = (\mathrm{flat}, S, \mathcal{O}, \Sigma)$. By the results in [29, 28, 13], it is directly related to Hecke algebras over $\mathcal{O}$ of certain weight two cuspidal eigenforms, cf. Theorem 2.8 below.

(d) Finally the choice of $\mathcal{O}$ is completely at our disposition due to the following result from [7], Appendix A.1: Under a change of rings $\mathcal{O} \to \mathcal{O}'$, the corresponding change of universal deformation rings is given by $\mathcal{R}_{\mathcal{O}',\Sigma} \cong \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{O},\Sigma}$, and similarly for the other rings we defined.

We formulate the following proposition, which explains the relation between $\mathcal{R}^l_{\mathcal{O},\Sigma}$ and $\mathcal{R}_{\mathcal{O},\Sigma}$.

Proposition 2.6 *Let Γ_Σ be the maximal abelian pro-l quotient of $G_{\mathbb{Q},\Sigma\backslash\{l\}}$. Then the following hold:*

(a) The group Γ_Σ is finite and the ring $\mathcal{O}[\Gamma_\Sigma]$ is isomorphic to the universal deformation ring of one-dimensional deformations of the trivial representation unramified outside $\Sigma\backslash\{l\}$.

(b) Let δ be a deformation of the trivial one-dimensional representation and $[\varrho]$ a deformation of $\bar\varrho$ whose determinant is of order prime to l away from l. The map on deformations $(\delta, [\varrho]) \mapsto [\varrho \otimes \delta]$ induces an isomorphism

$$\mathcal{O}[\Gamma_\Sigma] \otimes_{\mathcal{O}} \mathcal{R}^l_{\mathcal{O},\Sigma} \to \mathcal{R}_{\mathcal{O},\Sigma}\,.$$

Proof: Part (a) is from [22]. To construct an inverse of the map in (b), one needs $2 \neq l$ (more precisely $2 \in \mathbb{Z}_l^*$). We leave the details to the reader. ∎

Hecke algebras

Since $\bar\varrho$ is modular, a main source of deformations are Galois representations which arise from cuspidal eigenforms. Given a newform of level N which is a cuspidal eigenform, by results of Eichler, Shimura, Deligne and Deligne-Serre there is attached an l-adic Galois representation $\varrho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_f)$ which is unramified at primes not dividing lN and where K_f is a suitable finite extension of $\mathbb{Q}_l$, [10, 11]. With respect to a suitable basis, ϱ_f takes its values in $\mathrm{GL}_2(\mathcal{O}_f)$, where $\mathcal{O}_f$ is the ring of integers of K_f. The condition that ϱ_f is a lift of $\bar\varrho$, can be expressed in terms of congruences for the Hecke eigenvalues of f modulo the maximal ideal of $\mathcal{O}_f$. To single out the part of the Hecke algebra of the set of all modular forms of a given level and weight, which satisfy this congruence condition, one takes the completion at the corresponding maximal ideal.

The ramification behavior of ϱ_f finds its expression in the conductor $N(\varrho_f)$ of ϱ_f, cf. [15], § 3 for a definition. On the one hand one can directly compare the conductor of $\bar{\varrho}$ and of a deformation, loc. cit., Prop. 6. On the other hand, one has the following important result of Carayol linking N and $N(\varrho_f)$, [5]:

Theorem 2.7 *Let f be a cuspidal eigenform of prime-to-l level N. Then $N(\varrho_f)$ divides N with equality precisely when f is a newform away from l.*

Following Gouvêa, [15], it is natural to restrict the prime-to-l ramification of ϱ_f by bounding the conductor $N(\varrho_f)$, or equivalently by bounding the prime-to-l level N of f. Given Σ, we specify a prime-to-l conductor $N_\Sigma := \prod_{p \neq l} p^{n_\Sigma(p)}$, where, for each prime p different from l, the following defines $n_\Sigma(p) \in \mathbb{N}_0$:

(a) If $\bar{\varrho}_p$ is unramified, then we let $n_\Sigma(p)$ be 2, if $p \in \Sigma$, and 0 otherwise.

(b) If $\bar{\varrho}$ is p-ordinary, and $p \in \Sigma$, then $n_\Sigma(p) := n(\bar{\varrho}, p) + 1$.

(c) If $\bar{\varrho}$ is p-ordinary, and $p \notin \Sigma$, then $n_\Sigma(p) := n(\bar{\varrho}, p)$.

(d) If $\bar{\varrho}$ is ramified at p but not p-ordinary, then $n_\Sigma(p) := n(\bar{\varrho}, p)$.

See also [7], Sect. 5.1. Analogously, we define $N_{\Sigma'}$ for $\bar{\varrho}'$ and Σ'. We now define various Hecke algebras depending on Σ.

By $\mathcal{T}_{\mathcal{O},\Sigma}$ we denote the Hecke algebra of l-adic modular forms over $\mathcal{O}$ of prime-to-l conductor N_Σ, completed at the maximal ideal corresponding to $\bar{\varrho}$. We refer to [15] and [14] for more details. Note that one only considers the Hecke operators T_p for $p \nmid lN_\Sigma$ and the Diamond operators $\langle d \rangle$ for $(d, lN_\Sigma) = 1$, cf. [14], Thm. III.5.6. By $\mathcal{T}^l_{\mathcal{O},\Sigma}$ we denote the quotient of $\mathcal{T}_{\mathcal{O},\Sigma}$ corresponding to l-adic modular forms whose nebentype character away from l is of order prime to l. The definition of $\mathcal{T}'_{\mathcal{O},\Sigma'}$ is analogous.

In loc.cit., a deformation of $\bar{\varrho}$ to $\mathcal{T}_{\mathcal{O},\Sigma}$ is constructed by patching the Galois representations associated to classical cuspidal eigenforms. To make the patching work, the restriction $l \geq 7$ was made. But using the results of [6], due to Carayol and Serre, instead of [14], Thm. II.5.5, it suffices to assume $l > 2$, cf. [9], Sect. 5. Thus one obtains an induced map $\Phi_\Sigma \colon \mathcal{R}_{\mathcal{O},\Sigma} \to \mathcal{T}_{\mathcal{O},\Sigma}$. By its construction, $\mathcal{T}_{\mathcal{O},\Sigma}$ is generated by the traces of the Frobenii of the corresponding Galois representation. Therefore the map Φ_Σ is surjective. Similarly one has induced surjective maps $\Phi^l_\Sigma \colon \mathcal{R}^l_{\mathcal{O},\Sigma} \to \mathcal{T}^l_{\mathcal{O},\Sigma}$ and $\Phi'_{\Sigma'} \colon \mathcal{R}'_{\mathcal{O},\Sigma'} \to \mathcal{T}'_{\mathcal{O},\Sigma'}$.

Finally, let $\mathcal{O}$ be a discrete valuation ring which is a finite, totally ramified extension of $W(k)$ and which contains all eigenvalues for the Hecke operators T_n, n prime to lN_Σ, acting on the space of newforms of weight two and level dividing lN_Σ whose associated mod l representation is isomorphic to $\bar{\varrho}$. Let $\mathbb{T}$ denote the Hecke algebra over $\mathcal{O}$ on these forms, completed at its maximal ideal. In the case with strong supplementary conditions at l, a Hecke algebra $\mathcal{T}_\mathcal{D}$ was defined in [29], where $\mathcal{D}$ is as in Remark 2.5(c). We write $\mathcal{T}^s_{\mathcal{O},\Sigma}$ for this Hecke algebra. It is a quotient of $\mathbb{T}$, and finite flat over $\mathcal{O}$. As above, there arises a surjective ring homomorphism $\Phi^s_\Sigma \colon \mathcal{R}^s_{\mathcal{O},\Sigma} \to \mathcal{T}^s_{\mathcal{O},\Sigma}$.

The following identification between universal deformation rings and Hecke algebras was obtained as the central result in [29], [28] and [13].

Theorem 2.8 *Let $\mathcal{O}$ be as in the previous paragraph. Then Φ^s_Σ is an isomorphism between reduced complete intersection rings which are finite flat over $\mathcal{O}$. The* *localization of $\mathcal{T}^s_{\mathcal{O},\Sigma}$ at any minimal prime is isomorphic to the fraction field of $\mathcal{O}$.*

PROOF: The isomorphism statement in this form is [13], Theorem 1.1. The other parts are immediate from the construction of $\mathcal{O}$ and $\mathcal{T}^s_{\mathcal{O},\Sigma}$. ∎

Concerning the comparison of 'big' universal deformation rings and 'big' Hecke algebras, the following was conjectured in [15], p. 108.

Conjecture 2.9 For any Σ, Σ' the maps Φ^l_Σ, Φ_Σ and $\Phi'_{\Sigma'}$ are isomorphisms.

In the case that $\bar{\varrho}$ is modular, absolutely irreducible, unobstructed and attached to a modular form for $\Gamma_0(p)$ of non-critical slope for l, the proof of the above conjecture for Φ_Σ is one of the main results of [18].

3 Main results

For the remainder of this article, we assume that $\bar{\varrho}'$ satisfies the conditions given in Assumption 2.1 and that $\bar{\varrho}$ is its twist by ψ. In this section, we present our central results on the structure of big universal deformation spaces and the density of modular points in these spaces. The proofs will be given in Section 4 and 5, respectively. We also derive an isomorphism theorem for big universal deformation spaces and big Hecke algebras.

On the geometry of big universal deformation spaces

Theorem 3.1 *If $\bar{\varrho}$ is weakly l-ordinary, we assume that $\bar{\mu}_l \neq \bar{\chi}_l$. Then $\mathcal{R}^l_{\mathcal{O},\Sigma}$ is a complete intersection ring of dimension four. Furthermore, there exists a finite flat map $\beta\colon A := \mathcal{O}[[x_1,x_2,x_3]] \to \mathcal{R}^l_{\mathcal{O},\Sigma}$ such that one has an isomorphism*

$$\mathcal{R}^l_{\mathcal{O},\Sigma}/(\beta(x_1),\beta(x_2),\beta(x_3)) \cong \mathcal{R}^s_{\mathcal{O},\Sigma}\,.$$

Corollary 3.2 *Under the assumptions of the above theorem, the space X^l_Σ is equidimensional and has no embedded components.*

PROOF: Because any local complete intersection ring is Cohen-Macaulay, [21], p. 171, by the above theorem the ring $R := \mathcal{R}^l_{\mathcal{O},\Sigma}$ has this property. By [21], Thms. 17.4 and 17.6, it follows that the height of any associated prime $\mathfrak{p}$ of R is zero. So in particular, R has no embedded primes. Loc. cit. also implies that $\dim R = \dim R/\mathfrak{p} + \operatorname{height}\mathfrak{p}$ for any prime $\mathfrak{p}$ of R. Thus $\dim R = \dim R/\mathfrak{p}$ for all primes of height zero, and hence X^l_Σ is equidimensional. ∎

Definition 3.3 *A point P on X_Σ (or $X'_{\Sigma'}$) is called* modular *if there exists a cuspidal eigenform f such that*

(a) with respect to a suitable basis $\varrho_f\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_f)$ and $\bar{\varrho}_f = \bar{\varrho}$, and

(b) P is the kernel of the map $\mathcal{R}_{\mathcal{O},\Sigma} \to \mathcal{O}_f$ (respectively $\mathcal{R}'_{\mathcal{O},\Sigma'} \to \mathcal{O}_f$) corresponding to the deformation $[\varrho_f]$.

By L_i we denote the fields that arise when localizing $\mathcal{T}^s_{\mathcal{O},\Sigma}$ at minimal primes. (By our choice of $\mathcal{O}$, the L_i are isomorphic to the fraction field of $\mathcal{O}$.) The induced maps $\mathcal{R}^l_{\mathcal{O},\Sigma} \to L_i$ are denoted π_i, and the resulting points on X^l_Σ by P_i. From the definition of $\mathcal{T}^s_{\mathcal{O},\Sigma}$ and the construction of Φ^s_Σ, it follows that the P_i are modular points attached to an eigenform of l-level dividing l. With these notations in place, we can formulate the next important result on deformation spaces. Note that the word *dense* always means Zariski dense in this article.

Theorem 3.4 *Again we assume that $\bar{\mu}_l \neq \bar{\chi}_l$ in the case where $\bar{\varrho}$ is weakly l-ordinary. Let $\{X_j : j = 1, \ldots, c\}$ be the set of irreducible components of X^l_Σ. Then*

(a) There exists a dense open subset V of X^l_Σ containing all the P_i which is formally smooth over Spec $\mathcal{O}$.

(b) For each j the set $X_j \cap V$ is dense in X_j.

(c) Each X_j contains one of the points P_i.

Analogous results hold for X_Σ and $X'_{\Sigma'}$.

The key observation of the proof is the realization that the P_i are regular points on X^l_Σ. This can be seen as follows: By π_i we denoted the maps $R := \mathcal{R}^l_{\mathcal{O},\Sigma} \to L_i$. Let $\mathfrak{P}_i$ be the kernel of π_i and look at $R_{\mathfrak{P}_i}$. This is a local ring of dimension three with residue field L_i, and if we mod out by the three elements r_1, r_2, r_3, the result is L_i. Therefore these three must span $\mathfrak{P}_i/(\mathfrak{P}_i)^2$ which implies that the ring is regular. Having said all this, we postpone the proof to Section 4.

Corollary 3.5 *Under the assumption of the above theorem, the spaces X_Σ, X^l_Σ and $X'_{\Sigma'}$ are reduced.*

Proof: We only give the proof for X^l_Σ. For the other two cases, we refer the reader to the proof of Corollary 3.8 below, where the general principle is exposed. For simplicity, we write R for the coordinate ring of X^l_Σ.

Using the previous theorem, we choose for each component X_j of $\operatorname{Spec} R$ a smooth modular point P_j on it which meets no other component. Choose functions $f_j \in R$ that are zero on the components different from X_j, non-zero at P_j and such that $\operatorname{Spec} R_{f_j}$ is smooth over $\mathbb{Q}_l$. One easily checks that $f := \sum f_j$ is a non-zero divisor, and therefore that $R \hookrightarrow R_f$ is an inclusion. Since $\operatorname{Spec} R_f = \amalg \operatorname{Spec} R_{f_j}$ is smooth, the ring R_f contains no nilpotent elements, and hence neither its subring R. ∎

Remark 3.6 (a) The above results on big universal deformation spaces can be seen as a partial confirmation of some conjectures by Boston on the geometry of universal deformation spaces, [3]. Among other things they predict that X_Σ is a union of smooth four-dimensional components.

Our results are weaker than this. All irreducible components are reduced, generically smooth and of dimension four, they contain no embedded components and they are all flat over $W(k)$. On the other hand, maybe it is too much to hope for global smoothness, as there are examples of local non-smooth universal deformation rings, cf. [1], Thm. 6.2.

(b) Define c_Σ to be the number of components of X^l_Σ, and $d_\Sigma := \dim_K \mathcal{T}^s_{\mathcal{O},\Sigma} \otimes_{\mathcal{O}} K$. (The former number is rather mysterious, while the latter can be computed explicitly.) By Theorem 3.4 one has $c_\Sigma \le d_\Sigma$. Furthermore it is not difficult to see that c_Σ has to increase, i.e., that X_Σ has to 'grow' new components, whenever d_Σ increases. This raises the following question: Is the quotient d_Σ/c_Σ invariant under Σ, and if the answer is in the affirmative, what is its arithmetic significance?

On the density of modular points

Our main result here is the following, which will be proved in Section 5.

Theorem 3.7 *Suppose $\bar{\varrho}'$ satisfies the conditions of Assumption 2.1. Furthermore, if $\bar{\varrho}'_l$ is reducible, assume that $\bar{\mu}_l \neq \bar{\chi}_l$. Then for any Σ, the set of modular points on X^l_Σ whose prime-to-l conductor divides N_Σ is Zariski dense.*

Corollary 3.8 *Under the above hypotheses, for any set Σ (Σ'), the set of modular points of prime-to-l conductor dividing N_Σ (dividing $N_{\Sigma'}$) is Zariski dense in X_Σ (respectively $X'_{\Sigma'}$).*

Proof: We first prove the density of modular points in X_Σ. Recall that the relation between $\mathrm{Spec}(\mathcal{R}^l_{\mathcal{O},\Sigma})$ and $\mathrm{Spec}(\mathcal{R}_{\mathcal{O},\Sigma})$ is described by Proposition 2.6. Any closed point of $\mathrm{Spec}(\mathcal{R}_{\mathcal{O},\Sigma})$ can be obtained from a closed point of $\mathrm{Spec}(\mathcal{R}^l_{\mathcal{O},\Sigma})$ by twisting it with a character of Γ_Σ. We can twist modular forms by the same character, as the characters of Γ_Σ are of finite order. By [15], Prop. 6, the prime-to-l conductor of the twist is still bounded by N_Σ, and the density assertion for X_Σ follows from that of X^l_Σ.

We now turn to the case $X'_{\Sigma'}$ where we are given Σ'. Define Σ as the union of Σ' with all the primes p at which ψ'_p is non-trivial. Let $\tilde{\psi}\colon G_{\mathbb{Q}} \to W(k)^*$ be the Teichmüller lift of ψ where $\bar{\varrho} = \bar{\varrho}' \otimes \psi$. This is a character of finite order, and 'twisting by $\tilde{\psi}^{-1}$' induces an isomorphism from $\mathcal{R}_{\mathcal{O},\Sigma}$ to $\mathcal{R}'_{\mathcal{O},\Sigma'}$. As above one can twist modular forms by the same character. The definitions imply that $N_\Sigma = N_{\Sigma'}$, and the assertion for $X'_{\Sigma'}$ follows from that of X_Σ. ∎

The main consequence of the above theorem is the following comparison theorem of big universal deformation rings and big Hecke algebras.

Theorem 3.9 *Under the assumptions of Theorem 3.7, for any Σ, Σ', the maps $\Phi'_{\Sigma'}$, Φ_Σ and Φ^l_Σ, defined above Theorem 2.8, are isomorphisms. In other words, Conjecture 2.9 is verified in all these cases.*

Proof: The proof is only given for X^l_Σ. The generalizations to X_Σ and $X'_{\Sigma'}$ are obvious and left to the reader.

The scheme $\mathrm{Spec}(\mathcal{T}^l_{\mathcal{O},\Sigma})$ contains all modular points with prime-to-l level dividing N_Σ and character prime to l away from l. By Corollary 3.8, the closed subscheme $\mathrm{Spec}(\mathcal{T}^l_{\mathcal{O},\Sigma})$ is dense in X^l_Σ under the map induced by Φ^l_Σ. Density means that the kernel of the surjective map $\Phi^l_\Sigma\colon \mathcal{R}^l_{\mathcal{O},\Sigma} \twoheadrightarrow \mathcal{T}^l_{\mathcal{O},\Sigma}$ is nilpotent. By Corollary 3.5, the ring $\mathcal{R}^l_{\mathcal{O},\Sigma}$ is reduced, and so $\ker(\Phi^l_\Sigma) = 0$ as asserted. ∎

Remark 3.10 We now discuss a possible approach to obtain the above results under conditions more general than those of Assumption 2.1:

The theory of types, [7], §1, and recent results in [4], suggest to study the following program for any modular residual representation $\bar{\varrho}'$:

(a) Determine the 'simplest' type τ such that a twist $\bar{\varrho}_l$ of $\bar{\varrho}_l'$ arises as the mod l reduction of a modular representation ϱ_f of type τ.

(b) Determine the local universal deformation ring $\mathcal{R}_l^\tau$ for deformations of $\bar{\varrho}_l$ of type τ (and fixed determinant) as in [4].

(c) Prove a comparison theorem between small universal deformation rings and small Hecke algebras as in [7], where at l on imposes the condition that the deformations are of type τ.

Suppose this has been carried out successfully, and suppose $\mathcal{R}_l^\tau$ is a complete intersection, flat over $\mathcal{O}$ of relative dimension one. It is then easy to adapt the arguments in Sections 4 and 5 to obtain all results of this section also for $\bar{\varrho}$. (The properties of $\mathcal{R}_l^\tau$ are needed to prove the analogue of Lemma 4.2 for $\bar{\varrho}$, and (c) has to be used instead of Theorem 2.8.)

Using results from [7] and [4], it is indeed possible to work out the above program for certain $\bar{\varrho}$ which do not necessarily satisfy all of the conditions of Assumption 2.1. (One can do this for any $\bar{\varrho}$ which is the reduction modulo λ of $\varrho_{\pi,\lambda}$ as in [7], Cor. B.4.3, provided that the field F of loc. cit. has ramification degree at most $l-1$.) It will be interesting to further investigate the questions raised by the above program.

4 On the big universal deformation spaces

We start our investigation of X_Σ^l with the following theorem, which is a consequence of the results in [1]. It is the crucial link between the small and the big universal deformation rings.

Theorem 4.1 *Let $\mathcal{O}$ be the ring of integers of any finite totally ramified extension of the fraction field of $W(k)$. If $\bar{\varrho}$ is weakly l-ordinary, we assume that its adjoint character $\bar{\mu}_l$ is different from $\bar{\chi}_l$. Then the universal rings $\mathcal{R}_{\mathcal{O},\Sigma}$ and $\mathcal{R}_{\mathcal{O},\Sigma}^l$ are complete intersections, flat over $\mathcal{O}$ and of relative dimension three.*

If furthermore $\mathcal{O}$ is as in Theorem 2.8, then the kernel of the surjection $\mathcal{R}_{\mathcal{O},\Sigma}^l \to \mathcal{R}_{\mathcal{O},\Sigma}^s/(l)$ is generated by a regular sequence r_1, r_2, r_3, l of $\mathcal{R}_{\mathcal{O},\Sigma}^l$.

Proof: As this is not exactly what is proved in [1], we sketch a proof. To prove the assertions on $\mathcal{R}_{\mathcal{O},\Sigma}^l$, we introduce yet another universal deformation ring. Let $\eta: G_{\mathbb{Q},S} \to W(k)^*$ be the product of the Teichmüller lift of $\det \bar{\varrho}\bar{\chi}^{-1}$ and of χ. Let $\mathcal{R}_{\mathcal{O},\Sigma}^\eta$ denote the quotient of $\mathcal{R}_{\mathcal{O},\Sigma}$ which parametrizes lifts of determinant η. As all deformations parametrized by $\mathcal{R}_{\mathcal{O},\Sigma}^s$ have determinant η, this ring must be a quotient of $\mathcal{R}_{\mathcal{O},\Sigma}^\eta$.

Lemma 4.2 *The kernel of $\lambda: \mathcal{R}_{\mathcal{O},\Sigma}^\eta \twoheadrightarrow \mathcal{R}_{\mathcal{O},\Sigma}^s$ is generated by at most two elements, called r_1, r_2.*

Proof: The deformation theoretic conditions that describe the map λ are purely local at the prime l. Let η_l denote the local component of the character η at l. Denote by $\mathcal{R}_l$ the versal deformation ring of deformations of $\bar{\varrho}_l : D_l \to \mathrm{GL}_2(k)$ whose determinant is given by η_l.

We first discuss the case where $\bar{\varrho}$ is weakly l-ordinary. Define $\mathcal{R}_l^{\mathrm{ord}}$ as the quotient of $\mathcal{R}_l$ which parametrizes weakly l-ordinary deformations (of determinant η_l). Let $\tilde{\varrho} := \bar{\varrho}(\det \bar{\varrho})^{-1}$. By Lemma 2.4(a), it is weakly l-co-ordinary. The rings $\tilde{\mathcal{R}}_l$ and $\tilde{\mathcal{R}}_l^{\mathrm{co-ord}}$ will denote the versal rings for deformations and weakly l-co-ordinary deformations of $\tilde{\varrho}_l$, respectively, where in both cases it is assumed that the determinant be η_l^{-1}. By Lemma 2.4(a), there is a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_l & \twoheadrightarrow & \mathcal{R}_l^{\mathrm{ord}} \\ \downarrow\cong & & \downarrow\cong \\ \tilde{\mathcal{R}}_l & \twoheadrightarrow & \tilde{\mathcal{R}}_l^{\mathrm{co-ord}} \end{array}$$

where the vertical maps are isomorphisms. If $\tilde{\varrho}_l$ is l-co-ordinary, it is shown in [1], Cor. 7.4 and Thm. 8.1, that the kernel of $\tilde{\mathcal{R}}_l \twoheadrightarrow \tilde{\mathcal{R}}_l^{\mathrm{co-ord}}$ is generated by two elements, provided that $\bar{\mu}_l \neq \bar{\chi}_l$. The argument of loc.cit. applies to weakly l-co-ordinary deformations as well. Thus if $\bar{\mu}_l \neq \bar{\chi}_l$, the above diagram shows that there exist two elements, say r_1, r_2, which generate the kernel of $\mathcal{R}_l \twoheadrightarrow \mathcal{R}_l^{\mathrm{ord}}$.

In the flat case, it was shown in [26], that the local flat deformation ring is isomorphic to $\mathcal{O}[[X]]$, while $\mathcal{R}_l$ is isomorphic to $\mathcal{O}[[X_1, X_2, X_3]]$. Thus again, the kernel of the corresponding surjection is generated by two elements, which we call r_1, r_2 as above. To unify the notation, we write in either cases $\mathcal{R}_l^s$ for the quotient $\mathcal{R}_l/(r_1, r_2)$.

By the defining properties of $\mathcal{R}_l$, $\mathcal{R}_l^s$, $\mathcal{R}_{\mathcal{O},\Sigma}^{\eta}$ and $\mathcal{R}_{\mathcal{O},\Sigma}^{s}$, there is a push-out diagram

$$\begin{array}{ccc} \mathcal{R}_l & \longrightarrow & \mathcal{R}_{\mathcal{O},\Sigma}^{\eta} \\ \downarrow & & \downarrow\lambda \\ \mathcal{R}_l^s & \longrightarrow & \mathcal{R}_{\mathcal{O},\Sigma}^{s}. \end{array}$$

This shows that the kernel of λ is generated by the images of r_1, r_2 under $\mathcal{R}_l \to \mathcal{R}_{\mathcal{O},\Sigma}^{\eta}$, and thus the proof of the lemma is complete. ∎

Reducing modulo l the above lemma yields that

$$\mathcal{R}_{\mathcal{O},\Sigma}^{\eta}/(r_1, r_2, l) \cong \mathcal{R}_{\mathcal{O},\Sigma}^{s}/(l).$$

We now need the following result from obstruction theory whose proof will be given below. (Note that the g_i in the lemma below are allowed to be analytically dependent or even zero.)

Lemma 4.3 *There exists a positive integer n and elements $g_1, \ldots, g_{n-2}$ in the ring $\mathbf{R} := \mathcal{O}[[X_1, \ldots, X_n]]$ such that $\mathbf{R}/(g_1, \ldots, g_{n-2}) \cong \mathcal{R}_{\mathcal{O},\Sigma}^{\eta}$.*

By enlarging $\mathcal{O}$ if necessary, we assume that $\mathcal{O}$ is as in Theorem 2.8. This is compatible with the assertions made in Theorem 4.1, cf. Remark 2.5 (d). We conclude by Theorem 2.8, that $\mathcal{R}_{\mathcal{O},\Sigma}^{s}/(l)$ is a finite k-algebra. This point crucially uses the modularity of $\bar{\varrho}$.

Using the presentation given in Lemma 4.3, we find that $\mathbf{R}/(g_1, \ldots, g_{n-2}, r_1, r_2, l)$ is finite. This shows that the elements $g_1, \ldots, g_{n-2}, r_1, r_2, l$ must form a regular sequence in the regular local ring $\mathbf{R}$. In particular, the elements r_1, r_2, l form a regular sequence in the complete intersection ring $\mathcal{R}^{\eta}_{\mathcal{O},\Sigma}$. Hence $\mathcal{R}^{\eta}_{\mathcal{O},\Sigma}$ is flat over $\mathcal{O}$ and of relative dimension two.

As in Proposition 2.6, one shows that $\mathcal{R}^{l}_{\mathcal{O},\Sigma} \cong \mathcal{O}[[\Gamma]]\hat{\otimes}_{\mathcal{O}} \mathcal{R}^{\eta}_{\mathcal{O},\Sigma}$, where $\Gamma \cong \mathbb{Z}_l$ is the Galois group of the cyclotomic $\mathbb{Z}_l$-extension of $\mathbb{Q}$. As $\mathcal{O}[[\Gamma]] \cong \mathcal{O}[[x]]$ for some indeterminate x, we have proved that $\mathcal{R}^{l}_{\mathcal{O},\Sigma}$ is a complete intersection, flat over $\mathcal{O}$, of relative dimension three. Furthermore $\mathcal{R}^{s}_{\mathcal{O},\Sigma}/(l) \cong \mathcal{R}^{l}_{\mathcal{O},\Sigma}/(r_1, r_2, x, l)$, so that all assertions on $\mathcal{R}^{l}_{\mathcal{O},\Sigma}$ are shown.

The assertion on $\mathcal{R}_{\mathcal{O},\Sigma}$ follows from Proposition 2.6 upon noting that the ring $\mathcal{O}[\Gamma_\Sigma]$ is a complete intersection, finite flat over $\mathcal{O}$. ∎

Proof of Lemma 4.3: Let $\mathcal{R}_p$ denote the versal ring representing deformations of $\varrho_p\colon D_p \to \mathrm{GL}_2(k)$ which have fixed determinant η_p and are

(a) unramified, if p is not in S,

(b) ordinary (at p), if p is in $S\backslash\Sigma$,

(c) arbitrary, if p is in Σ.

Define $\mathcal{L}_p$ as the dual of the tangent space of $\mathcal{R}_p$, viewed as a subspace of $H^1(D_p, \mathrm{ad}^0_{\bar\varrho})$. The collection $\mathcal{L}$ of the $\mathcal{L}_p$ is the local set of deformation conditions corresponding to $\mathcal{R}^{\eta}_{\mathcal{O},\Sigma}$, cf. [2], Def. 4.2, or [12]. Furthermore for p any prime and $i \in \mathbb{N}_0$, we set $h^i_p := \dim_k H^i(D_p, \mathrm{ad}^0_{\bar\varrho})$. For $p \in \Sigma$, it is well-known that $h^1_p = \dim \mathcal{L}_p$, e.g. [22].

By [2], Cor. 6.4, or [12], Thm 2.39, there exists an *optimal* set of auxiliary primes S_{aux} for $\mathcal{L}$. Therefore, a presentation of $\mathcal{R}^{\eta}_{\mathcal{O},\Sigma}$ as a quotient of a ring $\mathcal{O}[[X_1, \ldots, X_n]]$ is given by [2], Thm. 5.6. Let I denote the corresponding kernel. The number n can be computed from [2], Lemma 5.5, as

$$n = -\dim H^0(D_\infty, \mathrm{ad}^0_{\bar\varrho}) + \sum_{p\in(S\cup S_{\mathrm{aux}})\backslash\{\infty\}} (\dim \mathcal{L}_p - h^0_p),$$

because $\mathrm{ad}^0_{\bar\varrho}$ and its Tate dual have trivial invariants. By [2], Lemma 5.5, or [12], Thm. 2.13, this expression can be rewritten as

$$n = 2 + \sum_{p\in S_{\mathrm{aux}}\cup\Sigma} h^2_p + \sum_{p\in S\backslash\Sigma} (\dim \mathcal{L}_p - h^0_p).$$

From the proof of [2], Thm. 3.8, or by direct computation, one can show that for any $p \in S\backslash\Sigma$ one has $\dim \mathcal{L}_p = h^1_p - h^2_p = h^0_p$. We find that $n = 2 + \sum_{p\in S_{\mathrm{aux}}\cup\Sigma} h^2_p$.

The presentation in [2], Thm. 5.6, also gives an estimate on the minimal number m of generators of I. It is bounded by the sum of the number of relations needed in a presentation of the $\mathcal{R}_p$ for $p \in (S \cup S_{\mathrm{aux}})\backslash\{\infty\}$. For p a prime in $S_{\mathrm{aux}} \cup \Sigma$, the ring $\mathcal{R}_p$ has a presentation with h^2_p relations, cf. [2], Thm. 3.8. For p in $S\backslash\Sigma$, no relations are needed, as the corresponding local rings are smooth. Thus $m \le \sum_{p\in S_{\mathrm{aux}}\cup\Sigma} h^2_p$, and the lemma is shown. ∎

Theorem 3.1 is now an immediate consequence of Theorem 4.1 and the following lemma:

Lemma 4.4 *Suppose R is a complete noetherian local $\mathcal{O}$-algebra of dimension $m+1$. Then elements $f_1, \ldots, f_m, l \in R$ form a regular sequence if and only if the induced map*

$$\beta\colon A := \mathcal{O}[[x_1, \ldots, x_m]] \to R : x_i \mapsto f_i$$

is finite flat.

Proof: This is a simple application of the Koszul complex. By [21], Thm. 22.3 (a)$\Leftrightarrow$(c), the ring R is flat over A if and only if $\mathrm{Tor}_i^A(R, k) = 0$ for all $i > 0$. Furthermore by [21], Thm. 16.5, the sequence $f_1, \ldots, f_m, l$ is regular in the ring R, if and only if the corresponding Koszul complex $K_\bullet(\{f_1, \ldots, f_m, l\}, R)$, cf. [21], p.127, has vanishing homology in all positive degrees.

The Koszul complex $K_\bullet(\{x_1, \ldots, x_m, l\}, A)$ is a resolution of k. (By the quoted result above, it is exact in positive degrees. In degree zero the homology can easily be identified with k.) Thus for any any A-module M one can compute $\mathrm{Tor}_i^A(M, k)$ by tensoring this complex with M and taking homology. We apply this to R. The definitions of the Koszul complex and of β imply that

$$K_\bullet(\{x_1, \ldots, x_m, l\}, A) \otimes_A R \cong K_\bullet(\{f_1, \ldots, f_m, l\}, R),$$

and hence $H_i(K_\bullet(\{f_1, \ldots, f_m, l\}, R)) \cong \mathrm{Tor}_i^A(R, k)$.

It only remains to verify the finiteness assertion on β. If the sequence $f_1, \ldots, f_m, l$ is regular, then the quotient $R/(f_1, \ldots, f_m, l)$ is of dimension zero and a k-algebra. Hence it has a finite basis over k, and by Nakayama's lemma R is finite over A. ∎

Our next aim is to prove Theorem 3.4. Using the sequence r_1, r_2, r_3, l defined above, we obtain the following diagram:

$$\begin{array}{ccc} A := \mathcal{O}[[r_1, r_2, r_3]] & \xrightarrow{\beta} & \mathcal{R}^l_{\mathcal{O},\Sigma} \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_R} \\ \mathcal{O} & \xrightarrow[\beta^s]{} & \mathcal{R}^s_{\mathcal{O},\Sigma}\,, \end{array}$$

where π_A and π_R are the quotients maps modulo (r_1, r_2, r_3). Let $b\colon X^l_\Sigma \to \operatorname{Spec} A$ be the map on spectra corresponding to β. Let $\Omega_{\mathcal{R}^l_{\mathcal{O},\Sigma}/A}$ denote the sheaf of differentials of $\mathcal{R}^l_{\mathcal{O},\Sigma}$ over A, and let $M := b_*\Omega_{\mathcal{R}^l_{\mathcal{O},\Sigma}/A}$, i.e. $\Omega_{\mathcal{R}^l_{\mathcal{O},\Sigma}/A}$ regarded as an A-module. As $A \to \mathcal{R}^l_{\mathcal{O},\Sigma}$ is finite, M is a finite A-module. By [25], Prop. 3.5 (a)$\Leftrightarrow$(b), the points $t \in \operatorname{Spec} A$ at which M vanishes are precisely those points at which the map $A \to \mathcal{R}^l_{\mathcal{O},\Sigma}$ is unramified.

Lemma 4.5 *The map $b\colon X^l_\Sigma \to \operatorname{Spec} A$ is unramified at all points above the point $\pi_A \in \operatorname{Hom}(\operatorname{Spec} \mathcal{O}, \operatorname{Spec} A)$, and so in particular, π_A is not in the support of M.*

PROOF: We need to show that b is unramified at all the points P_i. As b is of finite type, one can directly verify the definition for being unramified as given in [25], p. 21. For this, one has to show that the localization of $\mathcal{R}^s_{\mathcal{O},\Sigma}$ at any minimal prime is a finite separable field extensions of the fraction field of $\mathcal{O}$. This follows from Theorem 2.8. ∎

PROOF of Theorem 3.4: Let $U \subset \operatorname{Spec} A$ be the complement of the support of M, define $V := b^{-1}(U)$ and consider

$$\begin{array}{ccccc} \operatorname{Spec}(\mathcal{R}^s_{\mathcal{O},\Sigma}) & \xrightarrow{\pi_R} & V & \longrightarrow & X^l_\Sigma \\ \downarrow & & \downarrow{\scriptstyle b_{|V}} & & \downarrow{\scriptstyle b} \\ \operatorname{Spec}\mathcal{O} & \xrightarrow{\pi_A} & U & \longrightarrow & \operatorname{Spec} A. \end{array}$$

As M is finitely generated, the set U is open in $\operatorname{Spec} A$. By the previous lemma, it contains the points π_A so that it is non-empty. By the definition of M, the restriction $b_{|V}$ of b to V is unramified, and by the construction of V it is also finite flat. Therefore, [25], p. 22, the map $b_{|V}$ is a finite étale cover of U, and in particular smooth. Therefore its composition with the formally smooth map π_A is formally smooth, i.e., V is formally smooth over $\operatorname{Spec}\mathcal{O}$.

As b is finite and $\operatorname{Spec} A\backslash U$ has dimension at most three in $\operatorname{Spec} A$, the set $X^l_\Sigma\backslash V$ has dimension at most three. By Corollary 3.2, we know that X^l_Σ is equidimensional of dimension four. Hence $X_j \cap V$ is dense in X_j for all j.

Because $b_{|V}$ is finite étale, the sets $X_j \cap V$ must be disjoint and finite étale over U, too. Therefore the generic degree of $X_j \cap V$ over U is the same as the degree at the fiber over π_A, which shows that any set $X_j \cap V$ must contain a point P_i. As the sets $X_j \cap V$ are pairwise disjoint, this point cannot lie on any other $X_{j'}$. ∎

We now give the geometric content of Theorem 3.4 which will be important in Section 5.

Discussion 4.6 Let $\mathcal{R}_j$ be the coordinate ring of X_j. We fix one of the points P_i on X_j and call it Q_j. The corresponding residue field will be K_j, its ring of integers $\mathcal{O}_j$. Fix a surjection $\mathcal{O}[[y_1, \ldots, y_n]] \twoheadrightarrow \mathcal{R}_j$ for some n. When we base change $\mathcal{O}[[y_1, \ldots, y_n]]$ to K, the resulting ring is contained in the ring of convergent power series on the open rigid analytic n-ball over K. For a more rigorous discussion of the rigid space associated to $\mathcal{R}_j$, we refer the reader to [9], §1.1, or [20], §7.

Thus X_j/K can be viewed as a rigid threefold in this ball with the smooth point $Q_j/K = \operatorname{Spec} K_j$ on it. After possibly passing to a finite extension of $\mathcal{O}_j$, which we again denote by $\mathcal{O}_j$, there exists a closed rigid analytic 3-ball, call it B_j, around Q_j contained in X_j/K_j. To be explicit, one can choose coordinates of B_j such that its affinoid algebra A_j is given by

$$\Big\{ \sum_{(i_1,i_2,i_3)\in\mathbb{N}_0^3} a_{i_1,i_2,i_3} z_1^{i_1} z_2^{i_2} z_3^{i_3} : a_{i_1,i_2,i_3} \in K_j, |a_{i_1,i_2,i_3}| \to 0 \text{ as } i_1+i_2+i_3 \to \infty \Big\}.$$

In those coordinates, Q_j corresponds to the origin, and the K_j points of B_j are the elements of $\mathcal{O}_j \times \mathcal{O}_j \times \mathcal{O}_j$.

5 Mazur's infinite fern

Above we saw that, after rigidifying near the points Q_j, the space X^l_Σ/K looks like a rigid ball of dimension three. This is more or less the basic situation in [18] that led in a special case to the proof of the density of modular points in a universal deformation ring. We will closely follow this approach. A good introductory account of this can be found in [16], Sects. 7 and 8.

Definition 5.1 *Let $f(z) = \sum a_i q^i$ be the q-expansion of a normalized cuspidal eigenform (of some level and weight) with coefficients in $\bar{\mathbb{Q}}_l$. The* slope of f *is the number $v(a_l) \in \mathbb{Q} \cup \{\infty\}$.*

The slope of a normalized cuspidal eigenform f of weight k is called critical *if it is 0 or $k-1$.*

Remark 5.2 (a) Cusp forms with $\bar{\mathbb{Q}}_l$-coefficients of a given level and weight have a Hecke invariant basis with $\mathbb{Z}_l$ coefficients and form a finite dimensional vector space. Hence their Fourier coefficients are algebraic integers over $\mathbb{Z}_l$ and $v(a_l) \geq 0$ for any normalized cuspidal eigenform.

(b) The following remarks can be found in [17], Sect. 4:

If f is an l-newform of weight k for $\Gamma_0(l) \cap \Gamma_1(N)$, then its slope is $k/2 - 1$.

If f is an l-oldform in $\Gamma_1(lN)$, then there exists a cuspidal eigenform g for $\Gamma_1(N)$, such that $f(z)$ is in the span of $g(z)$ and $g(lz)$. If the slope α of f is different from $(k-1)/2$, then this span contains exactly one other normalized cuspidal eigenform f' of slope $k-1-\alpha \neq \alpha$. In particular α lies in $[0, k-1]$. The form f' is called the *twin* of f because of the following crucial observation made in loc.cit.: All eigenvalues of f and f' away from l are the same, and hence their associated Galois representations are identical.

Let N' be a positive integer prime to l. Let f be a cuspidal eigenform of weight k for $\Gamma_1(lN')$ for the Hecke algebra $\mathcal{T}_{N'}$ generated by the Hecke operators T_p for all primes p and Diamond operators $\langle n \rangle$ for all integers n prime to lN'. In particular this means that f is of a specific nebentype. By $\xi : (\mathbb{Z}/(lN'))^* \to \bar{\mathbb{Z}}_l^*$ we denote the corresponding nebentype character, and by $\xi_l : (\mathbb{Z}/(l))^* \to \mathbb{Z}_l^*$ its component at l. Let $\tau : (\mathbb{Z}/(l))^* \to \mathbb{Z}_l^*$ be the Teichmüller lift. We say that f *has l-character i*, if $\tau^i = \xi_l \tau^{-k}$ where $i \in \mathbb{Z}/(l-1)$.

For $a \in K$ and $r \in |K|$, let $B[a, r]$ denote the closed rigid analytic disc over K around a of radius r. For an affinoid ring R, let $\mathrm{Sp}(R)$ be the corresponding affinoid space. The following is an expanded version of [8], Thm. B5.7. and Lemma B5.3.

Theorem 5.3 *For any rational number $\alpha \geq 0$, positive integer k_0 and $i \in \mathbb{Z}/(l-1)$, there exist*

(a) a rational number $r \in |K|$,

(b) an affinoid ring R,

(c) a finite degree d cover $\mathrm{Sp}(R) \to B[k_0, r]$ of affinoid spaces,

(d) elements $a_n \in R$ with $|a_n| \leq 1$ for all $n \geq 1$,

such that the following holds. For any integer $k > \alpha + 1$ such that $|k - k_0| \leq r$ and any finite field extension L/K, there is a bijection between

(i) *classical cuspidal eigenforms for $\Gamma_1(lN')$ with respect to $\mathcal{T}_{N'}$ of slope α, weight k, l-character $k-i$ and Fourier expansion in L and*

(ii) *L-valued points x on $\mathrm{Sp}(R)$ in the fiber of $k \in B[k_0, r]$.*

The bijection is given by sending a point x to the cuspidal eigenform f_x with q-expansion $\sum_{n\geq 1} a_n(x)q^n$.

Furthermore, the affinoid $\mathrm{Sp}(R)$ parameterizes all overconvergent cuspidal eigenforms of slope α, level lN' and weight in $B[k_0, r]$, and R is the Hecke algebra of this family of overconvergent forms.

Proposition 5.4 *Let K_d be the span of all extension fields of K inside $\bar{\mathbb{Q}}_l$ of degree at most d. Then K_d is finite over $\mathbb{Q}_l$ and all classical cuspidal eigenforms parametrized by $\mathrm{Sp}(R)$ in the above theorem have Fourier expansion in K_d.*

Proof: The finiteness of $[K_d : K]$ is well-known. For the second assertion, note that K is the residue field at all the points $k \in \mathbb{Z} \cap B[k_0, r]$. Thus the fiber over k in $\mathrm{Sp}(R)$ is a finite commutative K-algebra of dimension at most d. Therefore all its closed points have degree at most d, and are hence contained in K_d. ∎

Theorem 5.3 explicitly specifies the behavior of the l-part of the Nebentype. The following lemma explains the behavior of nebentype characters away from l.

Lemma 5.5 *Suppose we are in the situation of Theorem 5.3. Take k_0 from there. For any $k \in \mathbb{Z} \cap B[k_0, r]$, there exists a positive rational number r' in the open interval $(0, r)$ such that the following holds: Given any two classical eigenforms f_1, f_2 parametrized by points x_1 and x_2 of $\mathrm{Sp}(R)$, of weights $k_1, k_2 > \alpha + 1$ in the disc $B[k, r']$. Suppose their Nebentype characters are ξ_1 and ξ_2. If $|x_1 - x_2| < r'$, then $\xi_1 = \xi_2$ away from l.*

Proof: The functions a_n are all rigid analytic on the affinoid space $\mathrm{Sp}(R)$. In particular given any finite number of such functions, any weight k and any $\varepsilon > 0$, one can find an r' such that whenever $x, x' \in \mathrm{Sp}(R)$ are over $B[k, r']$, then $|a_n(x) - a_n(x')| < \varepsilon$.

Choose primes $\{p_i\}_{i\in I}$ that form a system of representatives for $(\mathbb{Z}/(N'))^*$ and that are congruent to 1 modulo l. Via the formulae $\xi_j(p_i) = p_i^{1-k_j}(a_{p_i}^2 - a_{p_i^2})$ the functions a_{p_i} and $a_{p_i^2}$ completely determine all the values of the characters ξ_j. Choose r' as above for these functions and for $\varepsilon = |l|$.

Suppose there is an i such that $\xi_1(p_i) \neq \xi_2(p_i)$. As $\xi_1(p_i)/\xi_2(p_i)$ is an l-power root of unity, we have $|l| < |\xi_1(p_i) - \xi_2(p_i)|$. But due to our choice of r' and as $p_i \equiv 1 \pmod{l}$, we find:

$$\begin{aligned} |\xi_1(p_i) - \xi_2(p_i)| \;&\leq\; \max\{|(p_i^{1-k_1} - p_i^{1-k_2})(a_{p_i}^2(x_1) - a_{p_i^2}(x_1))|, \\ &\qquad |p_i^{1-k_2}(a_{p_i}^2(x_1) - a_{p_i^2}(x_1) - a_{p_i}^2(x_2) + a_{p_i^2}(x_2))|\} \\ &\leq\; \max\{|p_i^{1-k_1}(1 - p_i^{k_1-k_2})|, \varepsilon\} \;\leq\; |l|, \end{aligned}$$

a contradiction. ∎

We now recall [8], Cor. B5.7.1, which is obtained as a corollary of Theorem 5.3, where we add the observation of the previous lemma.

Corollary 5.6 *Let k_0 be an integer and $\alpha < k_0 - 1$. Suppose f is a cuspidal eigenform of weight k_0, slope $\alpha \neq (k_0-1)/2$, level lN' and Nebentype character ξ. Suppose f is new away from l and choose an i such that f has l-character $k_0 - i$. Then there exist an affinoid disk B' containing k_0 and rigid analytic functions a_n on B' such that the following holds. For any integer $k > \alpha + 1$ which lies in B' the expression*

$$\sum_{n\geq 1} a_n(k)q^n$$

is the q-expansion of a classical cuspidal eigenform f_k of weight k, slope α, level lN' and character ξ away from l and l-character $k - i$. Also, at $k = k_0$ we have $f_k = f_{k_0}$.

Furthermore, let A' be the affinoid algebra corresponding to B'. Then A' is the Hecke algebra of the family parametrized by B' via the maps a_n, and as before $|a_n| \leq 1$ for all n.

To fully understand the following proof, the reader is advised to read the article [18], as we quote various important results and techniques from loc.cit.

Proof of Theorem 3.7: We assume that the modular points are not Zariski dense. Let V be the smooth dense open subset given in Theorem 3.4 which is the disjoint union of the sets $X_j \cap V$. Thus there must be a j such that the modular points are not dense in $X_j \cap V$. We fix this j. Note that the set $X_j \cap V/K$ is dense in X_j. So we can find a function τ_j in the coordinate ring $\mathcal{R}_j$ which vanishes on all modular points and which is not identically zero on $X_j \cap V/K$. Let τ be the restriction of τ_j to the ball B_j around the point Q_j, in the notation of Discussion 4.6. This function cannot be zero, because B_j is Zariski dense in $X_j \cap V/K$. Also it vanishes on Q_j. Let f_j be the newform corresponding to Q_j and let N' be its prime-to-l level. By the discussion above Theorem 3.4, the function f_j has l-level dividing l, Hence Remark 5.2 (b) shows that f_j has finite slope.

The idea to prove density, cf. [18], is to look at the infinite fern as defined by Mazur in [24]. Say we call any family as in Corollary 5.6 a Coleman family. The infinite fern is the substructure in X^l_Σ which arises as the union of the images of all Coleman families. Even though p-adically this is like a one-dimensional curve, cf. [9], with respect to the Zariski topology, it will be shown to be dense in a suitable subvariety of X^l_Σ of codimension one, the Hodge-Sen-Tate null space, cf. [22].

Our first goal is to exhibit a good starting point for a Coleman family of overconvergent modular forms.

Lemma 5.7 *There exists a classical cuspidal eigenform of some weight $k \geq 2$ and level lN'', $N''|N'$, which is new away from l, has non-critical slope, and whose associated modular point lies in B_j.*

Proof: We first look at the case where the slope of f_j is zero. Let i be the l-character of f_j. If we would know that f_j has l-character 0 it had to be an old form, and we could replace it by its twin to achieve a situation where $\alpha > 0$, cf. Remark 5.2. But this may not be the case, and so we first need to apply Theorem 5.3 with $k_0 = 2$, $\alpha = 0$, $K = K_j$ and N'. Let x be the point on $\mathrm{Sp}(R)$ corresponding to

f_j. We choose a point $x' \in \mathrm{Sp}(R)$ over some integer $k > \alpha + 1 = 1$, $|k-2| < r$. By Lemma 5.5, and Theorem 5.3, we may also assume that the forms attached to x, x' have the same character away from l, by shrinking r if necessary. Then x' corresponds to a point P' on X^l_Σ. Shrinking r even further, if necessary, we may assume that x' corresponds to a point P' in the interior of B_j. Also we may take k of the form $k = 2 + l^n(l + i - 2)$ for $n \gg 0$ where we view $i \in [0, l-2]$. Then the l-character of $f_{x'}$ will be zero. Finally at this point we can replace the form by its twin (P' does not change), and now we are in the situation where the slope α is greater than 0.

If α is greater than zero, we go again through the previous argument. But we stop at the point where we reached a point in B_j, i.e., highly congruent to our given one, whose weight is bigger than $\alpha + 1$ and whose l-character is 0. If f_j is not new away from l, we replace it by the corresponding new form - this gives us N''. ∎

We replace Q_j by the point found by the previous lemma, and f_j by the corresponding form. In particular, f_j does not have critical slope and is a cuspidal eigenform for $\Gamma_0(N) \cap \Gamma_1(N'')$ of some weight $k > \alpha + 1$. Also, we replace K_j by $(K_j)_d$. Note that the radius of B_j did not shrink in all of this, but we do think now of the 'new' Q_j as the origin. The function τ still vanishes at Q_j.

Lemma 5.8 *Let f be a cuspidal eigenform of slope α, level lN' and weight k_0 which is new away from l and which corresponds to a point in B_j. Assume further that the weight of f is larger than $\alpha + 1$, and that f is defined over K, a finite extension of $\mathbb{Q}_l$. Then there exists a closed sub disk B'' over K around k_0 of the disc B' in Corollary 5.6 and a rigid analytic map $c\colon B'' \to B_j$ such that the following holds: For any point $k \in \mathbb{Z} \cap B''$, $k > \alpha + 1$ the image of the point $c(k)$ corresponds to the newform of slope α whose Fourier expansion is given by $\sum a_n(k) q^n$. Once B'' is chosen, the latter requirement makes such a map unique.*

Proof: By [9], Thm. 2.4.2 and Prop. 3.4.2, there is a continuous map $\Phi^{\mathrm{oc}} : \mathcal{T}^l_{\mathcal{O},\Sigma} \to A'$, which is induced from the natural map from overconvergent l-adic modular forms to l-adic modular forms. In Sect. 3.4 of loc.cit., it is assumed that N is square-free. As $\mathcal{T}^l_{\mathcal{O},\Sigma}$ is constructed from Hecke operators T_p for $p \nmid lN'$ and from the diamond operators, there is no problem in working with arbitrary N. But if desired, one can also generalize Sect. 3.4 of loc.cit. to arbitrary N.

Consider the diagram

$$\begin{array}{ccc} \mathcal{R}^l_{\mathcal{O},\Sigma} & \xrightarrow{\Phi^l_\Sigma \Phi^{\mathrm{oc}}} & A' \\ \beta \uparrow & \nearrow_{\gamma} & \\ \mathcal{O}[[X_1, X_2, X_3]], & & \end{array}$$

where $\gamma := \Phi^l_\Sigma \Phi^{\mathrm{oc}} \beta$. The rigid space associated to $\mathcal{O}[[X_1, X_2, X_3]]$ is the open three ball $B^3[0,1]$ of radius 1, cf. [20], Lemma 7.3.4. By continuity, the sections X_i must map to elements of A' of spectral norm less than 1, and thus γ factors through a closed three ball $B^3[0,r]$ of radius $r < 1$. As β is finite and flat, it is easy to see that $\Phi^l_\Sigma \Phi^{\mathrm{oc}}$ induces a map from $\mathrm{Sp}(A')$ to the rigid space associated to $\mathcal{R}^l_{\mathcal{O},\Sigma}$, but in fact also to the affinoid subspace Z sitting above $B^3[0,r]$. So one has a diagram of associated affinoid spaces

$$\begin{array}{ccc} Z & \longleftarrow & \mathrm{Sp}(A') \\ {\scriptstyle b}\downarrow & \swarrow & \\ B^3[0,r]. & & \end{array}$$

In a small neighborhood of the point π_A of $B^3[0,r]$, the map b is an étale cover, and all the points in the fiber of π_A are defined over $\mathcal{O}$ as well as local coordinate systems near these points. Let $P_f \in Z$ be the point in the fiber of π_A corresponding to f. Then the result follows by restriction to a suitable neighborhood of P_f. ∎

In analogy to [18], below Thm. 2, we call a curve C in B_j a modular arc, if C is the image of an overconvergent family over some closed disc $B[0,r]$ under a morphism as constructed above. Such an arc contains an infinite number of (classical) modular points. In particular if we let C be the image of B'' in B_j for the family through f given in the previous lemma, then C is such an arc. As C is not finite, it must be a curve. By shrinking B'', we may assume that C is smooth. As we want Q_j to be a point on C, we may have to change it again. If we wanted, we could shrink B_j to a neighborhood of Q_j such that $C \cap B_j$ looks like one of the coordinate axes with respect to suitable coordinates.

As in [18], one has a map $(X^l_\Sigma)_0 \times \Psi \to X^l_\Sigma$, where Ψ is the one-dimensional space of wild characters and $(X^l_\Sigma)_0$ the subspace of points x of X^l_Σ whose Hodge-Sen-Tate weight is of type $(0,y)$ for some y, cf. [18], p. 7. We have an induced subspace $(B_j)_0$ which contains C. By further shrinking the situation, we may assume that $(B_j)_0$ is disjoint from $(X^l_\Sigma)_{00}$, so that C does not contain the image of the weight zero points of B'. Let Ψ' be a neighborhood of Ψ near zero. Then $C \times \Psi' \to B_j$ is unramified, so that its image M is a smooth surface, which is in the zero locus of τ.

The affinoid algebra B_j is a unique factorization domain, so we can factor $\tau = \tau_1^{m_1} \cdot \ldots \cdot \tau_s^{m_s}$ where the τ_i are irreducible. The corresponding surfaces in B_j will be $M_i = \{\tau_i = 0\}$. We assume that $M = M_1$ near Q_j. If one of the M_i does not contain C, and hence intersect it in only finitely many points, we can shrink B_j and C, so that $\tau_i = 0$ is outside B_j. Thus we assume that C is contained in all M_i.

Let $C^{(\kappa)}$ denote the needles inside B_j of the infinite fern branching off C, cf. [18], Sect. 6. There are infinitely many of those, and they are again modular arcs. The argument in Sections 6 and 7 of loc.cit. then shows that M can only contain a finite number of needles. For this it is not important that B_j does not contain the whole image of $\Psi \times C$.

We choose a needle $C^{(\kappa)}$ that is not contained in M_1, and a smooth modular point Q'_j of finite non-critical slope on it. Furthermore, we replace B_j by a ball $B'_j \subset B_j$ around Q'_j whose radius is so small that M_1 is outside this ball. This yields the same situation as above, however the number of M_i that contain the intersection of this needle with B'_j is at most $s-1$. Inductively we arrive at a situation in which $s = 1$. Repeating the argument once more, we find a needle in B_j whose intersection with $\tau = 0$ is finite. This gives the desired contradiction, as any needle contains an infinite number of modular points of finite slope. ∎

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