Appendix 1: On the isomorphism $R_{\emptyset} \to T_{\emptyset}$

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In this appendix, using the notation of Sections 2 and 3 of the main article, we prove the following theorem:

**Theorem 1** Suppose for some auxiliary set of primes $Q$ one has the isomorphism $R_Q \to T_Q$ of finite flat $W(k)$-algebras. Then the canonical morphism $R_{\emptyset} \to T_{\emptyset}$ of minimal rings is an isomorphism as well.

We first need some simple preparatory results. Let $O$ be a discrete valuation ring which is finite flat over $W(k)$ and has residue field $k$. For a given residual representation $\overline{\rho}: G_Q \to \text{GL}_2(k)$ as in the main part of this article, let $X = (X_l)$ be a set of deformation conditions for deformations to CNL $O$-algebras as in [1], pp. 205 and 211, where $l$ runs over all the places of $Q$.

For example one could require that $X_p$ is the condition that the deformations considered are finite at $p$, or Selmer but not finite at $p$. For all but finitely many places, $X_l$ is the condition that the deformation is unramified at $l$. For a finite place $l \neq p$ at which ramification is allowed, $X_l$ could be the condition that arbitrary ramification is allowed. If $l$ is a prime in $Q$, where $Q$ is as in Section 2.1, then one could require that $X_l$ is the condition used in the definition of $R_Q^{\text{new}}$. If one fixes a lift $\eta: G_Q \to O^*$ of $\text{det} \overline{\rho}$, then a set of deformation conditions which includes this choice of determinant is denoted by $X^\eta$.

Furthermore, for any place $l$ of $Q$, one denotes by $R_{X,l}$ the versal deformation ring which parametrizes deformations of $\overline{\rho}_{|G_l}$ subject to the condition $X_l$. Also $R_X$ denotes the universal deformation ring which satisfies all the conditions of $X$. The existence of $R_X$ and $R_{X,l}$ is shown in [1], §3,4. Finally define $h^i(G_l, \text{Ad}(\overline{\rho})) := \dim_k H^i(G_l, \text{Ad}(\overline{\rho}))$ for $i \in \mathbb{N}_0$ and $l$ a place of $Q$, and similarly for $\text{Ad}^0(\overline{\rho})$ instead of $\text{Ad}(\overline{\rho})$. 
Define \( d \in \{0, 1\} \) and \( \text{Ad}_X(\overline{\rho}) \in \{\text{Ad}(\overline{\rho}), \text{Ad}^0(\overline{\rho})\} \) as \( d = 0 \) and \( \text{Ad}_X(\overline{\rho}) = \text{Ad}^0(\overline{\rho}) \) if \( X \) contains a fixed choice of determinant and \( d = 1 \) and \( \text{Ad}_X(\overline{\rho}) = \text{Ad}(\overline{\rho}) \) otherwise.

**Proposition 1** Suppose one is given \( \overline{\rho} \) and \( X \) as above. Let \( S \) be a finite set of places of \( \mathbb{Q} \) such that deformations of type \( X \) are unramified outside \( S \). Suppose further that

(a) For \( l \in S \setminus \{p, \infty\} \), the ring \( R_{X,l} \) is a complete intersection, flat over \( \mathbb{Z}_p \) and of relative dimension \( h^0(G_l, \text{Ad}_X(\overline{\rho})) - \Delta_l \) for some \( \Delta_l \in \mathbb{Z} \).

(b) The ring \( R_{X,p} \) is a complete intersection, flat over \( \mathbb{Z}_p \) and of relative dimension \( h^0(G_p, \text{Ad}_X(\overline{\rho})) + 1 + d - \Delta_p \) for some \( \Delta_p \in \mathbb{Z} \).

Let \( \Delta = \sum \Delta_l \) (which is defined because \( \Delta_l = 0 \) at all places of \( \mathbb{Q} \) where \( X \) does not allow ramification). Then \( R_X \cong \mathcal{O}[[x_1, \ldots, x_{n+d}]]/(f_1, \ldots, f_{n+\Delta}) \) for suitable \( n \in \mathbb{N} \) and \( f_i \in \mathcal{O}[[x_1, \ldots, x_{n+d}]] \).

**Proof:** By [1], Cor. 6.4, we can find a (not necessarily optimal) set of auxiliary primes \( S'_\text{aux} \), in the sense of op. cit. For \( l \in S' := S \cup S'_\text{aux} \cup \{p\} \), let \( R_{X,l} \cong \mathcal{O}[[x_{l,1}, \ldots, x_{l,n_l}]]/J_l \) be a presentation such that \( n_l = h^1(G_l, \text{Ad}_X(\overline{\rho})) \) for \( l \in S'_\text{aux} \) and \( n_l = \dim_k m_{R_{X,l}}/(p, m_{R_{X,l}}^2) \) for \( l \in S \cup \{p\} \), where \( m_{R_{X,l}} \) is the maximal ideal of \( R_{X,l} \). Let \( j_l \) denote the minimal number of generators of \( J_l \). By our assumptions on \( l \in S \cup \{p\} \) and because \( R_{X,l} \) is smooth over \( W(k) \) of relative dimension \( h^0(G_l, \text{Ad}_X(\overline{\rho})) \) for \( l \in S'_\text{aux} \), we have

\[
n_l = j_l + h^0(G_l, \text{Ad}_X(\overline{\rho})) - \Delta_l + \begin{cases} 0 & \text{for } l \in S' \setminus \{p\}, \\ 1 + d & \text{for } l = p. \end{cases}
\]

Let \( X' \) be the deformation problem with the same local constraints as \( X \) at primes \( l \) not in \( S'_\text{aux} \) and no local constraints at \( l \in S'_\text{aux} \), and \( m_{R_{X'}} \) the maximal ideal of \( R_{X'} \). Then by [1], Thm. 5.6, there exists a presentation \( R_X \cong \mathcal{O}[[x_1, \ldots, x_{n+d}]]/J \), where \( n + d = \dim_k m_{R_{X'}}/(p, m_{R_{X'}}^2) \) and \( J \) is generated by at most \( j := \sum_{l \in S'} j_l \) elements (all coming from the \( J_l \)). Because \( \overline{\rho} \) is odd, the formula in [1], Lem. 5.5(ii), yields

\[
n + d = d + \sum_{l \in S'} (n_l - h^0(G_l, \text{Ad}_X(\overline{\rho}))) = d - \Delta + \sum_{l \in S'} j_l,
\]

so that \( n + \Delta = j \), as asserted. \( \blacksquare \)
Suppose now that a global choice of determinant $\eta$ is fixed in the deformation problem $X^\eta$. If $X^\eta_p$ is the condition that the deformations are finite at $p$ or Selmer, then $\Delta_p = 0$, cf. [5], Sect. 2, except in the case when the restriction of $\overline{\rho}$ to a decomposition group at $p$ is decomposable and flat. This case is treated by the following lemma, which essentially follows from Ramakrishna’s thesis [7]:

**Lemma 1** Suppose the restriction $\rho|_{G_p}$ is isomorphic to $\left( \begin{array}{cc} \bar{\varepsilon} & 0 \\ 0 & 1 \end{array} \right)$, where $\bar{\varepsilon}$ is the mod $p$ reduction of the cyclotomic character $\varepsilon$. Let $X^\eta_p$ be the condition that the lifts are flat and of determinant $\varepsilon$. Then the versal deformation ring $R_{X^\eta,p}$ is isomorphic to $W(k)[[X_1, X_2]]$.

**Proof:** The functor of flat lifts of determinant $\varepsilon$ and of the form

$$\rho: G_p \to \text{GL}_2(R), g \mapsto \left( \begin{array}{cc} \varepsilon \chi(g) & b(g) \\ 0 & \chi^{-1}(g) \end{array} \right)$$

is representable and a versal hull of the functor in the lemma (note that by [4], Thm. 1.8(ii), any flat lift is reducible!). Therefore, it suffices to show that the universal ring for the latter be isomorphic to $W(k)[[X_1, X_2]]$. By [4], Lem. 3.4., the mod $p$ tangent space of the latter is of dimension 2, i.e., one has a presentation $R_{X^\eta,p} \cong W(k)[[X_1, X_2]]/I$ for some ideal $I$.

Let us assume $I \neq 0$. Write $m := (p)$ for the maximal ideal of $W(k)$ and consider the subspace

$$V^{\text{flat}} := \text{Hom}_{W(k)}(R_{X^\eta,p}, W(k)) \cong \{ (\alpha, \beta) \in m^2 \mid f(\alpha, \beta) = 0 \ \forall f \in I \}$$

of $m^2$. Let $\sigma_p$ be a Frobenius element in $G_p$ and denote by $\rho_{\alpha,\beta}$ the lift corresponding to the pair $(\alpha, \beta) \in V^{\text{flat}}$. The same convention is used for $\chi$ and $b$. Since $\chi$ can be expressed as a power series in $X_1, X_2$, the map

$$\phi: V^{\text{flat}} \to m : (\alpha, \beta) \mapsto \chi_{\alpha,\beta}(\sigma_p) - 1$$

can be represented by $h \in W(k)[[\alpha, \beta]]$. Let $|| : W(k) \to U := \{ p^{-n} : n \in \mathbb{N}_0 \} \cup \{ 0 \}$ be the normalized valuation on $W(k)$ and $I_p$ the ramification subgroup of $G_p$. Define

$$\psi: V^{\text{flat}} \to U : (\alpha, \beta) \mapsto |b(I_p)|,$$
where \(|b(I_p)| = \max\{|b(g)| : g \in I_p\}|. Using the results in [4], § 4.5, or in [7], one can show that \((\phi, \psi) : V_{\text{flat}} \to m \times U\) is surjective, i.e., given any character \(\chi\) and any \(n \in \mathbb{N} \cup \{\infty\}\), there exists a flat deformation to \(W(k)\) with the given character and \(|b(I_p)| = p^{-n}\).

Let now \(0 \neq f\) be an element of \(I\). Then

\[
X_f := \{(\alpha, \beta) \in \mathbb{C}_p^2 : |\alpha|, |\beta| \leq \frac{1}{p}, f(\alpha, \beta) = 0\}
\]

defines a possibly reducible one-dimensional affinoid variety. On each component, the analytic function \(h\) is either constant or has finite fibers. Therefore almost all the fibers of \(h\), considered as a function on \(X_f\), are finite. Because \(U\) is infinite and \((\phi, \psi)\) is surjective, all the fibers of \(h\) restricted to \(V_{\text{flat}}\) are infinite. Since \(V_{\text{flat}} \subset X_f\), the set \(V_{\text{flat}}\) must be the union of finitely many fibers of \(h\). This contradicts the surjectivity of \(h\) onto the infinite set \(m\). Hence we must have \(I = 0\) as asserted.

We claim that for \(l \in S \setminus \{p, \infty\}\) one has \(\Delta_l = 0\) in the following cases:

(i) \(X_l^n\) allows no ramification at \(l\), or

(ii) \(X_l^n\) imposes no local conditions at \(l\), or

(iii) \(\rho\) is ramified at \(l\) as in Section 2 and \(X_l\) is the condition that the deformation is minimal at \(l\), or

(iv) \(l\) is a prime in the set \(Q\) of Section 2, and \(X_l^n\) is the condition imposed for \(R_Q^\text{new}\) at \(l\).

Case (i) follows from [1], Thm. 2.4, case (ii) from [1], Thm. 3.8, and case (iii) from [1], Lem. 3.10 and Rem. 3.11. In case (iv), it is implicitly shown in [8], that \(R_{X^n,l} \cong \mathcal{O}[[T]]\) for some parameter \(T\), since clearly one has \(h^0(G_l, \text{Ad}^0(\bar{p})) = 1\) and it is shown in loc. cit. that any deformation to \(W(k)/(p^n)\) of type \(X_l^n\) can be lifted to a deformation to \(W(k)/(p^{n+1})\) of the same type. This yields the following corollary to Proposition 1:

**Corollary 1** Each of the rings \(R_\emptyset\) and \(R_Q^\text{new}\) has a presentation

\[
\mathcal{O}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)
\]

for suitable \(m \leq n\) and \(f_i \in \mathcal{O}[[x_1, \ldots, x_n]]\).
To further exploit the above corollary, we need the following result:

**Lemma 2** Suppose \( R \) is a CNL \( \mathcal{O} \)-algebra which is a finitely generated module over \( \mathcal{O} \). If \( R \) has a presentation \( R \cong \mathcal{O}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m) \) where \( m \leq n \), then \( R \) is a complete intersection, finite flat over \( \mathcal{O} \).

**Proof:** Let \( \pi \) be a uniformizing parameter of \( \mathcal{O} \). Because \( \mathcal{O} \) is a discrete valuation ring, the ring \( A := \mathcal{O}[[x_1, \ldots, x_n]] \) is a regular local ring of dimension \( n + 1 \). The given presentation of \( R \) shows that \( \bar{R} := R/(\pi) \) is a quotient of \( A \) by the ideal \( I := (\pi, f_1, \ldots, f_m) \). By the finiteness of \( R \) over \( \mathcal{O} \), the ring \( \bar{R} \) is a finite local ring over \( k \). Hence \( I \) is of height \( n + 1 \) and the Krull intersection theorem implies \( n = m \). Because \( A \) is regular local, by [6], Thm. 17.4, the elements \( \pi, f_1, \ldots, f_n \) form a regular sequence in \( A \). On the one hand this shows that \( f_1, \ldots, f_n \) is a regular sequence in \( A \), and therefore \( R = A/(f_1, \ldots, f_n) \) is a complete intersection, on the other it implies that \( \pi \) is a regular sequence of \( R \), i.e. that multiplication by \( \pi \) is injective. Since \( \mathcal{O} \) is a discrete valuation ring, \( R \) must be flat over \( \mathcal{O} \).

Combining Corollary 1 with Lemma 2 yields:

**Corollary 2** If \( R_Q \) is finite over \( \mathcal{O} \), then \( R_\emptyset \) is a complete intersection, finite flat over \( \mathcal{O} \).

For any \( \alpha \subset Q \), if \( R_Q^{\alpha-\text{new}} \) is finite over \( \mathcal{O} \), then it is a complete intersection, finite flat over \( \mathcal{O} \).

Note that in the proof of the first assertion, one uses that \( R_Q \to R_\emptyset \) is surjective.

**Proof of Theorem 1:** The following commutative diagram will be basic in our proof:

\[
\begin{array}{ccc}
R_Q & \cong & T_Q \\
\downarrow & & \downarrow \\
R_\emptyset & \longrightarrow & T_\emptyset
\end{array}
\]  

(1)

Due to the previous corollary and the standard fact that \( T_Q \) and \( T_\emptyset \) are finite flat over \( W(k) \), all the rings in the above diagram are finite flat \( W(k) \)-algebras. To show that the bottom vertical map is an isomorphism, it therefore suffices to prove this for a geometric point over the generic fiber of \( \text{Spec} W(k) \). Let \( \bar{K} \) be an algebraic closure of the fraction field \( K \) of \( W(k) \).
We introduce some notation. Let $M_Q$ denote the set of normalized eigen-forms for $\Gamma_0(l^\delta N(\rho)Q)$ such that their associated $l$-adic Galois representation $\rho_f$ has mod $l$ reduction isomorphic to $\bar{\rho}$. Analogously, one defines $M_\emptyset$ (with $Q = 1$). Finally, let $\rho_\emptyset : G_Q \to \text{GL}_2(R_\emptyset)$ denote a representation that represents the universal one.

Upon tensoring the above diagram over $W(k)$ with $\overline{K}$, one obtains

$$R_Q \otimes_{W(k)} \overline{K} \xrightarrow{\cong} T_Q \otimes_{W(k)} \overline{K} \xrightarrow{\cong} \prod_{f \in M_Q} \overline{K}$$

The isomorphisms on the right are a direct consequence of the definitions of $T_Q$ and $T_\emptyset$, and the fact that these rings are reduced (in the former case this is by choice of $Q$).

Suppose the map on the bottom left is not an isomorphism. Then there exists a modular form $f \in M_Q - M_\emptyset$ and a non-trivial morphism $\alpha : R_\emptyset \to \overline{K}$ such that the deformation class of $\rho_f$ is obtained from that of $\rho_\emptyset$ via the morphism $\alpha$. The definition of $R_\emptyset$ shows that $\rho_f$ must be unramified at all places dividing $Q$. But this forces the conductor of $f$ to be prime to $Q$ by [3] and contradicts the fact that $f \in M_Q - M_\emptyset$. ■

References


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