Forschungsseminar on Quaternion Algebras

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Abstract

The goal of the seminar is to obtain a thorough understanding of quaternion algebras and to get an idea about the construction of the Galois representation attached to a Hilbert modular form.

In the beginning the arithmetic theory of quaternion algebras will be introduced in detail and will be illustrated by many examples. Then Shimura varieties attached to quaternion algebras will be studied, as well as automorphic forms on them. Hilbert modular forms will be introduced and the Jacquet-Langlands correspondence will be stated. The final lecture will sketch the construction of the Galois representation attached to a Hilbert modular eigenform.

- Date: Thursday, 10-12 a.m., at the university.
- First session: 10 April 2008
- The language of the seminar is English.
- The webpage of the seminar is: http://maths.pratum.net/QuatAlg/
- Every participant should type some **text on his/her talk** containing at least precise definitions and statements of the theorems (that can be done after the talk). That text will be made available on the webpage.

Guest Lectures

1 (10/04/2008) Guest lecture by Ambrus Pal: The Manin constant of elliptic curves over function fields

We study the p-adic valuation of the values of normalized Hecke eigenforms attached to non-isotrivial elliptic curves defined over function fields of transcendence degree one over finite fields of characteristic p. Under certain assumptions we derive lower and upper bounds on the smallest attained valuation in terms of the minimal discriminant. As a consequence we show that the former can be arbitrarily small. We also use our results to prove for the first time the analogue of the degree conjecture unconditionally for infinite families of strong Weil curves defined over rational function fields.

2 (17/04/2008) Guest lecture by Marc-Hubert Nicole: Superspecial abelian varieties and their quaternion orders

Quaternion Algebras and their Arithmetic

For this part of the seminar we will essentially follow the standard reference [1].

3 (24/04/2008) Quaternion Algebras over fields, Marco Wolter

We define quaternion algebras over any field (as 4-dimensional central simple algebras), and then show the equivalences as in [9, Proposition 4.1.1]. For the proof we need the generally accepted reference on this topic, [1, Chapitre I], as well as [3, Chapter III §5].
Discuss reduced norm/trace (also the relation with the *usual* norm/trace for finitely generated algebras). Examples.
Skolem-Noether Theorem and its consequences ([1, Chapitre I §2]). The Theorem on the splitting field (Theorem 2.8) will be mentioned, *with* proof for global and local fields using [1, Lemme 3.1] - whose very simple proof should be included as well.

References: [1], [3], [9].

4 (08/05/2008) Arithmetic of Quaternion Algebras: Orders and Ideals, Marios Magioladitis

The basics on the arithmetic on quaternion algebras is introduced, as in [1, Chapitre I, §4]: (maximal) orders, (principal) ideals, (reduced) norm/discriminant; ideal classes (of an order $\mathfrak{o}$, in number $h(\mathfrak{o})$), order types (in number $t(\mathfrak{o})$). The inequality $t(\mathfrak{o}) \leq h(\mathfrak{o})$ will be proved.
Theorem 4.5 - the two-sided ideals of an order build an abelian group, freely generated by the prime ideals - must be proved.
If time permits, the paragraph on optimal embeddings could be also handled.

References: [1].

5 (15/05/2008) Quaternion algebras over local fields, Adam Mohamed

First we show the classification theorem for quaternion algebras over local fields (Théorème 1.1) - the many basic notions such as Hilbert symbol, valuations, etc., if at all, might be *briefly* recalled. Then we study maximal and Eichler orders in $M(2, K)$ - in particular, we prove that the maximal orders form a tree.

Reference: [1, Chapitre II, §§1–2].

6 (29/05/2008) Quaternion Algebras over global fields, Eduardo Ocampo

Define ramified (i.e. split) / unramified places, (totally) definite/indefinite quaternion algebra, (reduced) discriminant. Using the Minkowski-Hasse principle, prove the classification theorem for quaternion algebras over global fields (cf. [4, Theorem 2.7]). That for a given set of places (even in number) there exists (up to isomorphism) a unique quaternion algebra precisely ramified at those places can be shown in a particular case (as in [4, Proposition 2.10]) or in general using Hasse symbols and quadratic forms (cf. [3, Theorem 72:1]), depending on the speaker.
In case the global field is of class number one, the Proposition above presents no restriction - show this corollary, at least, for the rationals (see op.cit.).

References: [4] and [3].

7 (05/06/2008) Arithmetic of quaternion algebras over global fields, Björn Buth

We define orders and ideals, level, discriminant, type and class numbers. We begin by introducing the basic properties of orders as in [1, Chapitre III, §5.A]. The goal of this talk is to prove, using the adelic language, the finiteness of class number (hence of type numbers) and formulae relating them (cf. [1, pages 87–89]). For this we will use [1, Théorème Fundamental 4), page 61].

Reference: [1].

8 (12/06/2008) Trace formula and Brandt matrices, Ralf Butenuth

This talk concerns [1, Chapitre III, §5.C].
We study *optimal* embeddings of quadratic orders in quaternionic ones. We prove a trace formula in this case, [1, Théorème 5.11] and its consequences for Eichler orders, see loc.cit. For definite quaternion algebras over the rationals, there is a very nice exposition in [5, §1], which can be shown as well.

References: [1], [5], [7] and [6].

Shimura Curves, Automorphic Forms, Galois Representations

9 (19/06/2008) Shimura Curves I, Oscar Ledesma

Let B be a quaternion algebra over a totally real field F such that B is split at exactly one real place. Shimura curves are introduced as Riemann surfaces: they are the quotient of the upper half plane by the arithmetic (discrete) subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that one obtains by taking the elements of reduced norm 1 in a maximal order of B. Basic properties should be stated. The theory could be illustrated by giving examples of fundamental domains (for instance, with $F = \mathbb{Q}$). A brief mention of the moduli description in the case of $\mathbb{Q}$ could end the talk.

References: [1], [9], as well as references in the latter.

10 (26/06/2008) Shimura Curves II, Stefan Kukulies

The first aim of this talk is to study the adelic points of the Shimura curve: they will turn out to be a disjoint union of Shimura curves as studied in the previous talk. To be more precise, we again let B be a quaternion algebra over a totally real field F that is split at precisely one real place (although, for the definition of automorphic forms below, we could drop some of the assumptions). The contents of Section 3 of [21] should be explained (or the relevant part of [20]). In particular, we will need the projective tower of Shimura curves that one obtains by running through smaller and smaller compact open subgroups. It would be nice, if Deligne's (G, X)-language of Shimura varieties could be used; but, this language should not be introduced abstractly!

The final section of the talk should be devoted to the definition of quaternionic automorphic forms, both for a given level structure ([16] and [17], Section 2) and for the whole tower. Hecke operators should also be introduced. The Hecke action on the automorphic forms in the tower gives rise to automorphic representations of $(B \otimes_F \mathbb{A}_{F,f})^\times$; see [19] for analogies. The latter point should only be mentioned in passing.

References: [17], [21], [20], [19].

11 (03/07/2008) Hilbert Modular Forms, Phùng Hô Hai

This talk should introduce the adelic description of Hilbert modular varieties, Hilbert modular forms and Hecke operators on those. The corresponding automorphic representations should be mentioned briefly. It would again be nice, if Deligne's (G, X)-language could be used; again, this language must not be introduced abstractly. If time allows, the moduli interpretation could be mentioned.

References: [14], Sections 2-5 (and 7), [15].

12 (10/07/2008) Shimura Curves III, Kay Rülling

This talk is concerned with the Hodge theory of Shimura curves and is more difficult than the other talks. It should explain Sections 4 and 5 of [21] or the relevant parts of [20]. The aim is to explain the analogue of the Eichler-Shimura isomorphism for Shimura curves. It will be the main input in the sketch of the construction of the Galois representation in the subsequent talk. Another reference is [24].

13 (17/07/2008) The Galois Representation Attached to a Hilbert Modular Form, Gabor Wiese

The aim of this talk is piece together the input from the previous talks and to sketch the construction of the Galois representation attached to a Hilbert modular form, under some conditions. The talk should follow [21] or the relevant part of [20].

Briefly, it works as follows. Start with a Hilbert modular form over the totally real field F (under some restrictions) that is an eigenform for all Hecke operators. Pass to its automorphic representation.

Using the Jacquet-Langlands correspondence, which will be stated briefly, pass to an automorphic representation of $(B \otimes_F \mathbb{A}_{F,f})^\times$. This automorphic representation comes from a quaternionic modular form of some finite level. The Eichler-Shimura isomorphism from the previous lecture allows one to pass to an eigenclass in the H^1 of the Shimura curve (as complex variety) with a certain local system. At this point, we will believe the following: The Shimura curve has a model over F. The H^1 above can be compared with the étale H^1 of this model. On the latter, one now has commuting actions of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and the Hecke operators. This allows one to conclude that there is a 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on this étale H^1, corresponding to the system of Hecke eigenvalues of the Hilbert modular forms from the beginning. We will finish the seminar by stating that the characteristic polynomial of the Frobenius elements has the desired form, e.g. its traces are the Hecke eigenvalues. However, proving this would require a close analysis of the geometry of models of the Shimura curve of rings of integers, which would have to be the topic of a seminar on its own.

References: [21], [20], [23].

References

In order to retrieve these references, go to the seminar webpage (or the library).

Quaternion Algebras

[1] M.-F. Vignéras: *Arithmétique des Algèbres de Quaternions*

[2] K. Conrad: *A bibliography*

[3] O'Meara: *Introduction to quadratic forms over fields*

[4] Stefan Lumerell: *A description of quaternion algebras*

[5] Benedict Gross: *Heights and the special values of L-series*

[6] H. Shimizu: *On zeta functions of quaternion algebras*

[7] O. Körner: *Traces of Eichler-Brandt matrices and type numbers of quaternion orders*

Shimura Varieties and Shimura Curves

[8] Pete Clark: *Lectures on Shimura Curves*

[9] John Voight: *Shimura Curve Computations*

[10] David Kohel: *Hecke Module Structure of Quaternions*

[11] J.S. Milne: *Canonical Models of Shimura Curves*

[12] J.S. Milne: *Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles*

[13] J.S. Milne: *Introduction to Shimura varieties*

Quaternionic Modular Forms and Hilbert Modular Forms

[14] Gerard van der Geer: *Hilbert Modular Forms*

[15] Bas Edixhoven: *Hilbert Modular Forms and Local Langlands*

[16] B. H. Gross: *Algebraic Modular Forms*

[17] Lassina Dembélé: *Quaternionic Manin Symbols, Brandt Matrices and Hilbert Modular Forms*

[18] Kevin Buzzard: *Computing modular forms on definite quaternion algebras*

[19] *Forschungsseminar über automorphe Formen, Sommersemester 2007*

Galois Representations

[20] T. Saito: H*ilbert Modular Forms and p-Adic Hodge Theory*

[21] T. van den Bogaart: *Construction of Galois Representations in Cohomology of Shimura Curves*

[22] H. Carayol: *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*

[23] R. Taylor: *On Galois Representations Associated to Hilbert Modular Forms*

[24] P. Bayer, J. Neukirch: *On Automorphic Forms and Hodge Theory*