

Study group on p -adic L -functions and the Mazur-Tate-Teitelbaum conjecture

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Abstract

This seminar provides an introduction to the Mazur-Tate-Teitelbaum conjecture (MTT-conjecture). At the end of the seminar, the proof given by Orton (and based on earlier work of Darmon) is within reach – but due to time constraints will have to remain for independent study. There are many other proofs of the conjecture and, in fact, Orton’s proof is chronologically one of the most recent ones. A good survey over other proofs as well as alternative definitions of the \mathcal{L} -invariant is the article [Co1] together with our main reference [DT].

The first part of the seminar follows notes of a course given by Colmez, [Co0]. They contain basic results on L -functions and p -adic L -functions (for modular forms) as well as the measure theory needed in the construction of the latter. The notes will guide us up to the statement of the MTT-conjecture.

From there on we shall follow the notes [DT] by Dasgupta and Teitelbaum from the Arizona Winter School of 2007. A large part of these notes is devoted to the study of the p -adic upper half plane \mathcal{H}_p which can be thought of as an analog of the complex upper half plane \mathcal{H}_∞ . On it we shall study modular forms, learn how they are encoded in harmonic cochains and introduce Teitelbaum’s Poisson kernel as a useful technical tool. This background will help us to understand parts of Darmon’s theory of period integrals on $\mathcal{H}_p \times \mathcal{H}_\infty$. Having simultaneously p -adic and complex information will be crucial in the definition of the Orton- \mathcal{L} -invariant, as well as for Orton’s proof of the MTT-conjecture (which as pointed out above will not be given).

- Time: Thursday, 10-12 a.m.
- Place: T03 R04 D10
- Begin: 16 April 2009
- Language: English.
- Webpage (and gateway to bibliography): <http://maths.pratum.net/pAdicLF>
- Script: Every participant should type some **text on his or her talk** containing at least precise definitions and statements of the theorems (that can be done after the talk). That text will be made available on the webpage.
- Background: Some knowledge of the theory of modular forms.

Complex L -functions and periods

1 (16/04/2009) Complex valued L -functions, t.b.a

The talk should cover the following topics:

- (1) Analytic continuation to \mathbb{C} of L -functions arising from Mellin-transforms. (§1.1)
- (2) The case of the Riemann ζ -function. (§1.1)
- (3) Bernoulli numbers as special values of the Riemann ζ -function at odd negative integers. (§1.1)
- (4) The case of modular forms. (§3.1, 3.2)
- (5) The Euler product and functional equation of an L -function attached to a modular form. (§3.1, 3.2) and [DS, 5.9,5.10].

- (6) L -functions for elliptic curves from L -functions of modular forms (of weight 2). [MSD] or [DS, Thms.8.8.X].
- (7) The effect of twisting by characters (Cor. 3.3.2)

With so much material, this talk will be largely a survey talk. I definitely suggest to give the proof of (1) and explain how to show the analytic continuation of (2). Item (3) is quick if one is not too precise about estimates from analysis. For (4) one should simply state the standard bounds for the coefficients of the q -expansion of a modular form. (and recall the form of the q -expansion for arbitrary subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.) In (5) one could recall how the Euler product is a natural consequence of the identity $\sum T_p^n X^n = \frac{1}{1 - T_p X + p^{k-1} T_{p,p} X^2}$ (for a form of weight k and a prime p not dividing the level of the form) for the generating function for Hecke operators together with $T_{mn} = T_m T_n$ for relatively prime $m, n \in \mathbb{N}$. If one finds the time to give a second proof (and not just sketch of proof) in this talk, it might be that of the functional equation.

The all important tool to study the L -function of an elliptic curve is the conjecture of Taniyama-Shimura. Without there is no known proof of the entireness of the L -function of an elliptic curve (or the existence of a functional equation). For elliptic curves over \mathbb{Q} the conjecture is a theorem due to Breuil-Conrad-Darmon-Taylor following groundbreaking work of Wiles. In the present talk, not much can be done beyond stating one or two versions of the Taniyama-Shimura theorem (and what it means for an elliptic curve to be modular).

Literature: All references starting with § are to [Co0].

2 (23/04/2009) The periods of a newform, t.b.a.

The main results to be presented are the definition of the periods Ω_f^\pm of a newform f and the algebraicity of the renormalized special values at 1 of the L -function of twists of f . Some background can be found in §2, in particular in §2.5 and §2.6. The main result is Thm. 3.3.3. whose prove is given in §3.3. An alternative proof of Theorem 3.3.3 is sketched in [Co1, §1]. It gives less precise information but seems conceptually clearer. Therefore it is recommended to first give the key ideas of this latter proof before giving the proof of [Co0, §3.3] which uses the Rankin-Selberg method.

For newforms f of weight 2 with rational Fourier coefficients one should relate the periods Ω_f^\pm to the periods Ω_E^\pm of the corresponding strong Weil elliptic curve E , e.g. [MSD], [Co2, §0.2], [Si, Si2].

Literature: All references starting with § are to [Co0].

Functions and distributions on \mathbb{Z}_p and p -adic L -functions

For any motive M one has a simple recipe by which one attaches a complex L -function to M , defined as an Euler product whose local terms come from étale realizations of the motive. It is conjectured that this Euler product, which initially only converges on a right half plane, extends to an analytic function $L(M): \mathbb{C} \rightarrow \mathbb{C}$ defined on the entire complex plane except for possibly finitely many poles. Moreover standard conjectures predict a functional equation. In contrast to this, for arbitrary motives there is only a highly conjectural and sophisticated method, largely due to Perrin-Riou, to associate with M an analytic p -adic L -function $L_p: \mathbb{Z}_p \rightarrow \mathbb{C}_p$. We make no attempt to understand Perrin-Riou's framework but content ourselves in studying the special cases of p -adic L -functions associated with the Riemann ζ -function and to modular forms. We will follow the classical construction of p -adic L -functions which are characterized by an interpolation property for certain special values coming from the complex L -function. The interpolation property can either come from the density of sets of the form $-(p-1)\mathbb{N}+c$ in \mathbb{Z}_p (for $c \in -\mathbb{N}$) or the density of locally polynomial functions (of bounded degree) in certain spaces of functions $\mathbb{Z}_p \rightarrow \mathbb{C}_p$. The key step in the construction of the p -adic L -function is the construction of a suitable measure μ (associated with f), so that $L_p(s)$ is, essentially, given by $\int x^s \mu$, the Mazur-Mellin transform. This

procedure is formally very similar to the definition of the Mellin transform used to define complex L -functions (of modular forms).

3 (30/04/2009) Continuous functions and measures on \mathbb{Z}_p , t.b.a.

This talk should cover §1.2–1.4 of [Co0]. It contains largely preparatory material for the later study of p -adic L -functions. Important results are Theorem 1.3.2 on the coefficients of the Mahler expansion, Theorem 1.4.5 on the coefficients of the Amice transform. It is suggested that also parts of §1.4.2 are presented. One should keep in mind that all one is doing here is to describe a Banach space (that of continuous functions on \mathbb{Z}_p) and its dual. Since we like to think in terms of bases and sequences, the Mahler expansion and the counterpart by Amice are enormously useful. I think there is enough time to give the proofs of the main results of this section.

4 (07/05/2009) The Leopoldt-Kubota p -adic ζ -function and more functions, t.b.a

Having all the background from the previous talk, the first half of the present talk should cover §1.5 of [Co0]. Important are the results on the p -adic ζ -function from §1.5 (Def. 1.5.10, Thm. 1.5.7) due to Kubota and Leopoldt. The p -adic ζ -function expresses congruences among the Bernoulli numbers and among the twisted special values at 1. Further background on p -adic ζ -functions from a slightly different and perhaps more elementary perspective can be found in [Ko, §2]. In particular, it contains a good explanation for the factor $\frac{1}{1-\omega(a)^{1-k}\langle a \rangle^{1-s}}$ in [Co0, Def. 1.5.10].

For the p -adic L -functions of modular forms, the Banach spaces introduced in Talk 3 are not sufficient. The second half of the present talk introduces further function spaces. It should cover §1.6 and §1.7.1. It might be useful to start with a survey of what is to come, i.e., with an overview, as given for instance in §1.10. Since the functions in §1.6 do not (seem to) play such an important role in what we do later, the discussion of §1.6. should be kept short, e.g., one could present subsection 1.6.1 as a survey and skip completely the proof of the main result Theorem 1.6.3 (and thus subsection 1.6.2).

In the remaining time, I suggest to fully present subsection 1.7.1 on analytic functions on discs. Its content is basic in the definition of locally analytic functions. It would be nice if much could be proved.

Literature: Except for [Ko, §2], all references are to [Co0].

5 (14/05/2009) Locally analytic functions and distributions, t.b.a

The aim is to complete the introduction of the function and distribution spaces that are to be found in the summary §1.10 with the properties given there **and** to give a detailed proof of Theorem 1.9.7. The latter will be the main tool in the construction of p -adic L -functions. The material is 1.7.2–1.10.

Some suggestions: In the previous talk we learned the definition of analytic functions. This class needs to be extended to so-called locally analytic functions, cf. §1.7.2. Note that **all** integrands we shall later consider are locally analytic! It would be good to present a sizable part of the proof of Theorem 1.7.8. Clearly also the duals of the spaces in §1.7, i.e., distributions (of some order) discussed in §1.8.1, will play a key role. Theorem 1.8.4 should be stated and explained – perhaps also via the examples in 1.8.2. The proof could be skipped.

Theorem 1.9.1 should be easy to prove (if we take earlier results for granted). Theorems 1.9.2 and 1.9.3 will not be needed (they underline, however, the importance of the spaces \mathcal{C}^r). After giving Definition 1.9.4 one should see what is needed to prove 1.9.7 and do it. One certainly will need the characterizations in Theorem 1.9.5. (I think in the assertion of Theorem 1.9.7 there is a slight imprecision: Namely the range of j is $0 \leq j \leq N$ and not $j \in \mathbb{N}$.)

Literature: All references are to [Co0].

6 (28/05/2009) The p -adic L -functions of modular forms and elliptic curves, t.b.a.

The construction of the p -adic L -function of a newform should be given in detail, following §3.4. One first obtains a measure and then by integration the p -adic L -function. For the proof given in §3.4, one may have to recall various results on Hecke operators, periods, twists etc. (from Chapters 2 and 3 of [Co0] and from the previous talks).

The Mazur-Tate-Teitelbaum conjecture is explained and described at the end: Conjecture 3.4.7. It would be nice to explain how to deduce the special case for the p -adic L -function of an elliptic curve. One may have to consult [MTT] or [Co2, §03-04]. An overview of this talk is given in [Co1, §2-4].

The case of elliptic curves over \mathbb{Q} has the appealing feature that the constant \mathcal{L} can be explicitly described!, see [MTT, II.1]. Probably this was the first case in which numerical experiments were possible.

The true nature of the \mathcal{L} -invariant may best be explained using either Fontaine theory and the p -adic local Langlands correspondence, cf. [DT, 4.1].

Literature: References starting with § are to [Co0].

Analysis on the p -adic upper half plane \mathcal{H}_p

The rest of the seminar leads toward Orton's proof of the MTT-conjecture (following earlier work of Darmon). The main reference will be [DT].

7 (04/06/2009) The p -adic upper half plane, t.b.a

The talk should cover [DT, Ch. 1] except for 1.3.7: introduce the p -adic upper half plane \mathfrak{X} , explain the way it is a rigid analytic space and describe its reduction map. As a secondary reference the book [FvdP] might be helpful – the first edition!! For our applications, it suffices to explain affinoid and rigid analytic subspaces of \mathbb{P}^1 .

This talk is either something for someone who knows already a lot about the p -adic upper half space or who wants to learn a lot about it. The preparation will have to include quite a number of things not explained in [DT, Ch. 1].

8 (18/06/2009) Morita duality, t.b.a.

This talk consists of three parts of which the last is presumable the longest: In the first part the notion of locally analytic function needs to be extended from \mathbb{Z}_p to $\mathbb{P}^1(K)$ and $GL_2(K)$ (and subgroups of the latter) for K a p -adic (locally compact!) field. I suggest to be less abstract than [DT, 2.1.1]. For instance one can obtain useful (disjoint coverings) of the above two spaces from the surjections $\mathbb{P}^1(K) \cong \mathbb{P}^1(\mathcal{O}) \twoheadrightarrow \mathbb{P}^1(\mathcal{O}/\pi^n)$ and $GL_2(K) \twoheadrightarrow GL_2(K)/(1 + \pi^n M_2(\mathcal{O}))$ whose images are discrete. One could even assume $K = \mathbb{Q}_p$.

The second part should introduce $\mathcal{O}(k)$ from [DT, 1.3.7] for $k \in \mathbb{Z}$ even, explain

$$0 \longrightarrow P_{k-1} \longrightarrow \mathcal{O}(2-k) \xrightarrow{\left(\frac{d}{dx}\right)^{k-1}} \mathcal{O}(k) \longrightarrow H_{dR}^1(\mathfrak{X}) \longrightarrow 0,$$

introduce C^{an} and C^{la} (I suggest to rename them to C^{lan} and C^{lag}), and to explain for $k \leq 0$ the sequence

$$0 \longrightarrow C^{\text{lag}}(K, k)/P_{-k} \longrightarrow C^{\text{lan}}(K, k)/P_{-k} \xrightarrow{\left(\frac{d}{dx}\right)^{1-k}} C^{\text{lag}}(K, 2-k) \longrightarrow 0.$$

The main part should be the statement and proof of the Morita equivalence, Theorem 20. (Note that by functional analysis $\mathcal{O}(k)''_b$ is reflexive, i.e., the strong bidual of $\mathcal{O}(k)$ is isomorphic to $\mathcal{O}(k)$.) For the proof of Theorem 20, define I_k , show it is well-defined, introduce residues and prove surjectivity. (Probably this is too optimistic.)

9 (25/06/2009) The Poisson kernel and harmonic cochains, t.b.a.

Another important isomorphism is given in Theorem 25, where the Poisson integral is shown to be the transpose of the map I_k from the previous talk. Theorem 25, as well as Corollary 26, should be stated and proved.

Then harmonic cochains on the Bruhat-Tits tree should be introduced. The residue map sends then $\mathcal{O}(k)$ into $C_{\text{har}}(k)$. It would be nice if much of the important Theorem 29 could be proved. In particular this theorem computes the kernel of the residue map.

In the following section §2.3 it is proved that the restriction of the residue map to bounded distributions is an isomorphism. Perhaps the proof of Theorem 30 could be skipped. The consequences, Corollary 31 and Theorem 32 should (perhaps again without proof) be explained.

The main application of the correspondence of §2 for bounded distributions is Proposition 33. Here one starts with a quaternion algebra B over \mathbb{Q} which is assumed to be ramified at ∞ but not at p . Let D be an order of B over $\mathbb{Z}[1/p]$, let M be prime to the discriminant of B and $\Gamma(M)$ the level M -congruence subgroup of the group of units of D of reduced norm one. A summary of the theory of p -adic uniformization is given in the middle of page 28 of [DT]. For us the main consequence is that the residue map defines a Hecke equivariant isomorphism $\mathcal{O}(k)^{\Gamma(M)} \rightarrow C_{\text{har}}(k)^{\Gamma(M)}$.

\mathcal{L} -invariants

10 (02/07/2009) Modular symbols and Teitelbaum's \mathcal{L} -invariant, t.b.a.

This talk has a lengthy description which attempts to motivate some definitions and concepts that will be important in the remaining talks. The audience will have to judge the success of this attempt.

Motivation for the integrals at the bottom of [DT, p.31]: Let the notation be as at the beginning of [DT, §3.1]. Suppose, for motivational purposes that the form F from loc.cit. is a newform of weight 2, has Hecke eigenvalues in \mathbb{Q} and corresponds via the Jacquet-Langlands correspondence to a classical modular form. Then there exists an elliptic curve E over \mathbb{Q} with the same L -function as F . In the same way that E/\mathbb{C} is a factor of the Jacobian of $X_0(pMN)$, the curve E/\mathbb{C}_p is the factor of the Jacobian of $\Gamma(M)\backslash\mathcal{H}_p$ defined by F .

Observe that since the p -part of the conductor of F is p , the elliptic curve E has split multiplicative reduction at p . Hence over \mathbb{C}_p it has a Tate uniformization $E(\mathbb{C}_p) \cong \mathbb{C}_p^*/q^{\mathbb{Z}}$ for some $q \in \mathbb{C}_p^*$ with $|q| < 1$. Therefore it is natural to ask about a rigid analytic map from $\Gamma(M)\backslash\mathcal{H}_p$ to $\mathbb{C}_p^*/q^{\mathbb{Z}}$ which induces the above-mentioned algebraic map from the Jacobian of $\Gamma(M)\backslash\mathcal{H}_p$ to E/\mathbb{C}_p . Such a map can be given as follows (see [BDG, §2]):

Because F is of weight 2, it defines a $\Gamma(M)$ -invariant harmonic cocycle $c_F: \text{Edges}(\mathcal{T}) \rightarrow \mathbb{C}_p$. By multiplying F with a suitable scalar, one may assume that c takes its values in \mathbb{Z} (and so that the image spans some \mathbb{Z}). By the theory of the Poisson kernel, there exists a measure γ_F on $\mathbb{P}^1(\mathbb{Q}_p)$ such that

$$F(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\lambda_f(t).$$

Since c is integer-valued, so is μ_f . This allows one to define for $\tau_1, \tau_2 \in \mathcal{H}_p$ an integral

$$\int_{\tau_1}^{\tau_2} f(z) dz := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-\tau_1}{t-\tau_2} \right) d\lambda_f(t).$$

Defining also $\int_{\tau_1}^{\tau_2} f(z) dz := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log \left(\frac{t-\tau_1}{t-\tau_2} \right) d\lambda_f(t)$, one has the following key result (see [BDG, §2]):

Theorem 1 (1) $\forall \gamma \in \Gamma(M), \tau \in \mathcal{H}_p: \int_{\tau}^{\gamma\tau} f(z) dz \in q^{\mathbb{Z}}$.

(2) For fixed $\tau_1 \in \mathcal{H}_p$, the map $\tau \mapsto \int_{\tau_1}^{\tau} f(z) dz$ induces the map $\text{Pic}^0(\Gamma(M)\backslash\mathcal{H}_p) \longrightarrow \mathbb{C}_p^*/q^{\mathbb{Z}}$.

(3) $\text{ord} \left(\int_{\tau_1}^{\tau_2} f(z) dz \right) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \text{ord} \left(\frac{t-\tau_1}{t-\tau_2} \right) d\lambda_F(t) = \sum_{e: v(\tau_1) \rightarrow v(\tau_2)} c_F(e)$, where the second equality holds for τ_i above vertices of \mathcal{T} under the reduction map and where the sum is over all edges from the initial to the terminal vertex.

(4) $\log \left(\int_{\tau_1}^{\tau_2} f(z) dz \right) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log \left(\frac{t-\tau_1}{t-\tau_2} \right) d\lambda_F(t)$.

In the notation on page 31, the expression in (3) and (4) with $(\tau_1, \tau_2) = (\tau, \gamma\tau)$ are called $h_{\text{ord}}^F(\gamma, 1)$ and $h_{\log}^F(\gamma, 1)$, respectively. As a corollary to the above one deduces:

Corollary 2 $\forall \tau \in \mathcal{H}_p, \gamma \in \Gamma(M) : \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log \left(\frac{t-\gamma\tau}{t-\tau} \right) d\lambda_F(t) = \frac{\log(q)}{\text{ord}(q)} \int_{\mathbb{P}^1(\mathbb{Q}_p)} \text{ord} \left(\frac{t-\gamma\tau}{t-\tau} \right) d\lambda_F(t)$.

One can readily verify that the assignments

$$\gamma \mapsto \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log \left(\frac{t-\tau}{t-\gamma\tau} \right) d\lambda_F(t), \quad \gamma \mapsto \int_{\mathbb{P}^1(\mathbb{Q}_p)} \text{ord} \left(\frac{t-\tau}{t-\gamma\tau} \right) d\lambda_F(t)$$

define elements of $H^1(\Gamma(M), \mathbb{C}_p) = \text{Hom}(\Gamma(M), \mathbb{C}_p)$. Let $H^1(\Gamma(M), \mathbb{C}_p)^F$ denote the subspace on which the Hecke algebra acts in the same way as on F . Then

Theorem 3 The subspace $H^1(\Gamma(M), \mathbb{C}_p)^F$ is 1-dimensional (over \mathbb{C}_p), it contains both of the above cocycles and the cocycle for ord is a basis.

The generalization of the above theorem to arbitrary Hecke eigenforms forms and of arbitrary weight is [DT, Thm 36]. Teitelbaum's definition of the \mathcal{L} -invariant $\mathcal{L}_{\mathcal{T}}$ is the analog of the above corollary. If the above is explained, not much should be said about [DT, Thm 36], though. However, to give the present talk it is recommended to familiarize oneself with the above concepts – but not to give proofs.

The next part of the talk should introduce modular symbols as a means to describe classical modular forms, following [DT, p. 33]. Note that M and N are *redefined* in [DT, §3.2]; they satisfy $M = pN$ and the pair (M, N) would have been called (pMN, MN) in the previous section of [DT]. Let D be the divisors on $\mathbb{P}^1(\mathbb{Q})$, i.e., the free abelian group on elements of $\mathbb{P}^1(\mathbb{Q})$ which should be thought of as the boundary of \mathcal{H}_{∞} . Let D_0 be the kernel of the degree map $D \rightarrow \mathbb{Z}$. To any newform $f \in S_k(\Gamma_0(M))$ one can attach two modular symbols

$$\varphi_f^{\pm} \in \text{Hom}_{\mathbb{Z}}(D_0, \text{Hom}_{\mathbb{C}}(P_{k-2}(E), E))^{\Gamma_0(M)} \quad (1)$$

as in [DT, p. 33], where E is the finite extension of \mathbb{Q} containing all Hecke eigenvalues of all eigenforms of $S_k(\Gamma_0(M))$. It might be helpful to relate the right hand side of (1) in a second way to the space of cusp forms, namely by stating formula [DT, (13)] and the Eichler-Shimura isomorphism [DT, (15)]. It is also important to connect Teitelbaum's definition to [Co0, §3.3].

One of the useful features of modular symbols (which in some way have come up in Talk 6 already) is that they provide closed expressions for the algebraic special values $L^{\text{alg}}(f, \chi, j)$, cf. [DT, Lem. 40]. These are the values that were interpolated to define the p -adic L -functions. (There is no need to discuss much of this, since this was done in Talk 5.)

11 (16/07/2009) Harmonic modular symbols and the \mathcal{L} -invariant of Orton, t.b.a.

The aim of this talk is the definition of Orton's \mathcal{L} -invariant. The talk starts by introducing Darmon's mix of p -adic harmonic cochains and modular forms and symbols. This mix will be important to define the \mathcal{L} -invariant $\mathcal{L}_{\mathcal{O}}$ in a way analogous to $\mathcal{L}_{\mathcal{T}}$ and to prove the MTT-conjecture. All of §3.2.2 should be explained. It would be nice if a proof of Prop. 42 could be indicated. It is not that long, but one has to look up [Or, §2.1] or [BDG, Prop. 3.2]. Using now [DT, Prop. 43], one can associate with every cusp form for $\Gamma(M)$ which is p -new two (times two) cohomology classes in $H^1(\Gamma_0^p(N), \mathcal{M})$ as in formulas [DT, (10), (11)].

To properly define the \mathcal{L} -invariant, in the remainder of the talk one should cover [DT, 3.3.1–3.3.2]. Some key intermediate results are Lemma 45 and Corollary 55 (whose proof

depends on Theorem 54). Their combination constitutes the analog of Theorem 3 above. Note that in even at the end of 3.3.2, Orton's invariant is not properly defined, because Corollary 55 requires equation (20) which is only proved in §3.3.3.

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