Prof. Dr. Gebhard Böckle, Dr. J. M. Cerviño 18. Oktober 2011

Working group seminar WS2011/12 :

Borcherds products

Wednesdays from 9:15 to 10:45, INF 368, room 248[1]

The goal of the seminar is to understand the fundamental work of R. Borcherds on infinite product expansions of holomorphic automorphic forms on certain orthogonal groups (resp. on Grassmannians). Probably the most prominent and well known example of product expansion of a modular form is

$$\Delta(\tau) = q\prod_{n>0}(1-q^n)^{24} =: \Psi_{f_0}(\tau).$$

The theory of Borcherds products gives a conceptual explanation of this very particular example. This modular form corresponds to the Borcherds product associated with the holomorphic modular form $f_0(\tau) := 12\cdot\theta_{\mathbb{Z}}(\tau) = \sum_{n\geq 0} c_0(n)q^n$ (theta series of the standard rank one lattice, hence of weight $1/2$). Namely, the weight of Ψ_{f_0} is $c_0(0) = 12$, and the exponents of the factors $(1-q^n)$ are given by $c_0(n^2)$ (all of which equal 24, in this simple case).
Another example involves the j-function. We have

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots = q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\cdots$$

The exponents now correspond to (3 times) the coefficients of a nearly holomorphic modular form $f_1 = \sum_{n\gg\infty} c_1(n)q^n = q^{-3} - 248q + 26752q^4 + \ldots$ of weight $1/2$ for $\Gamma_0(4)$.

This theory has its importance justified already by the beauty of this conceptual construction of *modular products*. As one may know (has heard), it has several applications to other branches of mathematics, which we may not be able to pursue in this seminar (Monstrous Moonshine, generalized Kac–Moody algebras, unimodular lattices, mirror symmetry, arithmetic geometry).
In regard to the last application, Borcherds products allow one to construct (vector bundle) sections with prescribed divisors (on Grassmannians). The latter, e.g., was used in the construction of the second example above. The "3 times" corresponds to the fact that the j-function vanishes to order three at $(-1+\sqrt{-3})/2$, while Ψ_{f_1} vanishes to order $c_1(-3) = 1$, so $\Psi_{f_1}^3 = j$. The leading factor q^{-1} (as well as q^1 in the previous example) has an additional explanation; cf. [Bo1, pages 42–43] for more details.

The main reference for Borcherds products is the article [Bo2]. We will mainly concentrate on understanding this paper. The basic ingredient is the theory of the Howe correspondence, or better said, of the Harvey–Moore's extension of it to modular forms with poles – to be referred as *"the singular Howe correspondence"*.
We want to mention, that the construction of these products was given for the first time in the previous paper [Bo1], with a somewhat more involved proof of a less general result than in [Bo2]. The two approaches are different in nature. In the first one, the main effort is put on the proof that a given product is modular (an automorphic form) and that it has a meromorphic continuation, whereas in the second, Borcherds starts with an automorphic form and then proves that it has a product expansion (essentially, that it is the logarithm of another automorphic form). In this second and more general approach, the singular Howe correspondence *"defines"* the alluded

[1] Begin: October 19.

automorphic form – so the great contribution, from the point of view of an *outsider*, is to give a precise description of the Fourier expansion of this automorphic form as well as of the divisor it defines on Grassmannians. The concrete form of the singular Howe correspondence in our case (corresponding to the dual reductive pair $(\widetilde{\mathrm{Sp}}_*, O(*,*))$) was already existent: classically in the original papers of Howe, Gelbart and Rallis: [Ho1], [Ge] and [Ra] (a must reference for the talks involved with it!); the regularized version in the paper of Harvey and Moore [HM].

We now turn to the description of the seminar talks, not without making the disclaimer, that none of us is an expert or whatsoever in this topic – so we apologize for possible inaccuracies and/or not being able to make a smoother path into this exciting jungle. We hope that this last sentence makes sense to the participants (the organizers included!) – in the best case, after the seminar is finished.

At different stages the habilitation thesis of J. H. Bruinier, [Br] might be of help.

Please: Do complete the mentioned *goal(s)* of each lecture. Either it/they are concretely described in the explanation of each lecture, or the goal is simply the content of the explanation itself. In case one goal is explicitly mentioned, the emphasis of the talk must be put on it (the rest, if any, should be done as well – this, being part of a well prepared lecture.
Any citation without a particular reference to an article is from [Bo2].

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1. Vector valued modular forms, some Fourier transform calculations

Explain [Bo2, §2], the examples (in detail, except 2.3 and less from 2.4, if you are not familiar with Kohnen's plus space) and prove Lemma 2.6. Further, prove Lemma 3.2.

Literature: [Bo2].

Speaker: Yamidt Bermudez–Torbon 26.10.

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2. Siegel modular forms

This should be a crash lecture on Siegel modular forms. Define the symplectic group $\mathrm{Sp}_n(\mathbb{R})$, Siegel upper half space $\mathbb{H}_n$ and the action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$. Define then a *Siegel modular form*, show its Fourier expansion and give examples (Eisenstein series, theta series associated with lattices).
Literature: Follow closely [Koh]. As standard reference we have as well Freitag's book [Fr]. For theta series of indefinite quadratic forms, look at [Sie] and [Vign].

Speaker: Ann–Kristin Juschka 02.11.

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3. Siegel theta functions

Start by recalling (/introducing?) Siegel theta series for indefinite quadratic forms (cf. [Sie], [Vign]). The goal of this lecture is to reproduce the proof of [Bo2, Thm. 4.1], as well as of its corollary.
Literature: [Bo2], and for the basics in theta series of indefinite forms [Sie], [Vign].

Speaker: Tommaso Centeleghe 09.11.

The next few lectures concern the Howe correspondence. We might have simply defined the function Φ as done in §6 without further comments – this would have been a sin! The theory of Howe correspondences is, as already mentioned, the main tool used in [Bo2]. Even though for proving Borcherds main result we only need its concrete form (i.e. the definition of Φ), the theoretical framework is an *important piece of mathematics*.
The first talk sets up basic definitions, such as the metaplectic group and the Weil representation. The second introduces the concept of dual pairs, explains the duality correspondence for the case $(\widetilde{\mathrm{Sp}}_*, O(*,*))$ and gives some further local examples. The third and fourth talk study in detail the Howe correspondence for the mentioned reductive pair.

4. Weil representation

Present the whole of sections 1 and 2 from [Pr] – Heisenberg group, metaplectic group, Schrödinger representation, Weil representation, and the models for these.
Literature: [Pr].

Speaker: Yujia Qiu 16.11.

5. Dual pairs and correspondence, local examples

The goal of this talk is to understand at least Sections 3 and 5 of part of [Ge]. These sections correspond to Sections 3 and 4 in [Pr], which may/should be used as the main reference (with further literature therein).
Literature: [Ge], [Pr].

Speaker: Narasimha Kumar 23.11.

6. and 7. Howe duality I and II

These two talks form a block. One should start with §0 of [Ra1], explain all basic definitions, the global Weil representation (cf. paragraph (VII) of Section 0 loc.cit.), θ_ϕ, et cetera. In short, all Section 0 and the statement (without proof) of Theorem I.1.1. The second talk explains Section 2 with proofs, at least until Remark I.2.3 (and maybe orally the statement of Theorem I.2.2, cf. later work on the local Howe correspondence for reductive pairs with one compact group, where Howe proves its validity over an archimedean place: [Ho3] and [Ho2]).
Literature: [Ra1], [Ho3], [Ho2].

Speaker(s): David Guiraud, Hendrik Verhoeck 30. and 07.12.

We continue with our main reference [Bo2]. The next step is to study *in detail* [Bo2, §6] (complete proofs!). After this, we present Theorems 5.2 and 5.3, the first one with proof.

8. The singularities of Φ

Explain in great detail Section 6 – it could take more than 90 minutes, but must take less than 130.
Literature: [Bo2].

Speaker: Andreas Maurischat 14.12.

9. The Fourier expansion of Φ (I)

Present Theorems 5.2 and 5.3, the former with proof.
Literature: [Bo2].

Speaker: Ralf Butenuth 21.12.

10. The Fourier expansion of Φ (II)

Proof of the main theorem [Bo2, Thm. 7.1]. In the remaining time (?) mention (or prove) the short lemma *"the embedding trick"*.
Literature: [Bo2].

Speaker: Patrik Hubschmid 11.01.

The remaining talks explain how the results can be used to construct holomorphic *"modular"* infinite products.

11. Holomorphic infinite products (I)

Explain the beginning of Section 13. Prove the lemmas and formulate Theorem 13.3!
Literature: [Bo2].

Speaker: Juan Cerviño 18.01.

12. Holomorphic infinite products (II)

Please give handout for the formulation of the Theorem 13.3 and maybe for examples to be presented. This talk finishes Section 13 (and the seminar).
Literature: [Bo2].

Speaker: Juan Cerviño 25.01.

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