On $p$-adic Hecke algebras for $GL_2$ over totally real fields

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0. Introduction

The purpose of this paper is to lay the foundation of the theory of $p$-adic Hecke algebras for $GL_2$ over totally real fields. Apart from the flatness of the ordinary part of the universal Hecke algebra over the Iwasawa algebra, almost all results obtained in our previous papers [12] and [14, §1] in the case where $F = \mathbb{Q}$ are generalized to arbitrary totally real fields $F$. Our result holds without any exception for the prime $p$; thus, the assumption: $p \geq 5$ which we made in [12] and [14] is now eliminated. One peculiar feature in the treatment of Hilbert modular forms is the existence of multiple weight modular forms; i.e., those forms with the automorphic factor: $\prod c_\sigma(z_\sigma + d_\sigma)^{k_\sigma}$ for mutually distinct $k_\sigma$'s. In order to guarantee the stability of the space of integral cusp forms under Hecke operators, we have to modify a little in Section 3 the definition of Hecke operators $T(\pi)$ unless $k$ is parallel (i.e. $k_\sigma = k_\tau$ for all $\sigma, \tau$). When we consider the congruence subgroups of the adelized $GL_2$ over $F$ of type

$$\left\{ x \mid x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod p^a \right\},$$

the independence of the $p$-adic Hecke algebras relative to the weight $k$ no longer holds. We can recover the isomorphism between the ordinary Hecke algebras of different weights $k$ and $l$ only when the difference of $k$ and $l$ is parallel. Thus we have infinitely many distinct Hecke algebras parametrized by the classes of weights modulo parallel ones. The presentation of this phenomenon is one of the motives of this work. Besides the obvious generalizations to totally real fields of the results obtained for $\mathbb{Q}$ in [14] on Galois representations, we hope to discuss in a future occasion how to unify these infinitely many Hecke algebras into the universal one and also to discuss an intimate relation between our results and the theory of cyclotomic $\mathbb{Z}_p$-extensions over $\mathbb{Q}$. In fact, the construction of Galois representations over our big Hecke algebra of parallel weight (i.e. in the case of $v = 0$; see below for details) has already been done by Wiles [40]. His result even covers the totally real fields of even degree.
We shall now give a sketch of our result in the simplest case of odd \( p \)-power level. We fix throughout the paper a rational prime \( p \) and a totally real field \( F \) of finite degree over \( \mathbb{Q} \). Let \( I \) be the set of all embeddings of \( F \) into \( \mathbb{R} \). The module \( \mathbb{Z}[I] \) of weights of \( F \) is by definition a free module generated by the elements of \( I \). Each weight \( k = \sum_{\sigma} k_{\sigma} \cdot \sigma \) can be considered as a quasi character of \( F^\times \) assigning \( x^k = \prod_{\sigma} (x^{k_{\sigma}})^{\sigma} \) to \( x \in F^\times \). We fix an algebraic closure \( \overline{\mathbb{Q}}_p \) of the \( p \)-adic field \( \mathbb{Q}_p \) and take the algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \) inside \( C \). We fix once and for all an embedding: \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \), and hence any algebraic number in \( \overline{\mathbb{Q}} \) can be considered as a complex number as well as a \( p \)-adic number in \( \overline{\mathbb{Q}}_p \). Let \( \Phi(k) \) be the subfield of \( \overline{\mathbb{Q}} \) generated by the values \( x^k \) for all \( x \in F \), and let \( \mathcal{O}(k) \) denote the valuation ring of \( \Phi(k) \) corresponding to the embedding: \( \Phi(k) \hookrightarrow \overline{\mathbb{Q}}_p \). In \( \mathbb{Z}[I] \), there is one specific element \( t = \sum_{\sigma} t_{\sigma} \cdot \sigma \) corresponding to the norm map: \( F^\times \to \mathbb{Q}^\times \). We write \( \xi \sim \eta \) (resp. \( \xi \geq \eta \)) for two elements \( \xi, \eta \in \mathbb{Z}[I] \) if \( \xi - \eta \in \mathbb{Z} \cdot t \) (resp. \( \xi_{\sigma} - \eta_{\sigma} \geq 0 \) for all \( \sigma \in I \)). We shall fix one class of \( \mathbb{Z}[I]/\mathbb{Z} \cdot t \) and take the smallest non-negative representative \( v \in \mathbb{Z}[I] \) of this class. For each \( 0 \leq n \in \mathbb{Z}[I] \) with \( n = -2v \), we put \( k = n + 2t, w = v + k - t \) and \( \dot{w} = t - v = k - w \). Let \( H \) be the upper half complex plane. We consider the following automorphic factor:

\[
\det(\gamma)^{-\dot{w}} j(\gamma, z)^k = \prod_{\sigma \in I} (a_{\sigma} d_{\sigma} - b_{\sigma} c_{\sigma})^{w_{\sigma} - 1} (c_{\sigma} z_{\sigma} + d_{\sigma})^{k_{\sigma}}
\]

for \( \gamma = \begin{pmatrix} a_{\sigma} & b_{\sigma} \\ c_{\sigma} & d_{\sigma} \end{pmatrix} \in \text{GL}_2(\mathbb{R})^I \) and \( z = (z_{\sigma})_{\sigma} \in H^I \). Let \( F_\Lambda \) (resp. \( F_{\infty}, F_f \)) be the adele ring of \( F \) (resp. the infinite part of \( F_\Lambda \) and the finite part of \( F_\Lambda \)). Then \( \text{GL}_2(F_\infty) \) can be identified with \( \text{GL}_2(\mathbb{R})^I \), and its connected component \( \text{GL}_2^+(F_\infty) \) with the identity acts naturally on \( H^I \). Let \( C_{\infty,+} \) be the stabilizer in \( \text{GL}_2^+(F_\infty) \) of \( z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in H^I \). We denote by \( \mathbb{Z} \) the integer ring of \( F \) and put \( \hat{\mathbb{Z}} = \mathbb{Z} \otimes \mathbb{Z} \), where \( \hat{\mathbb{Z}} \) is the product of the \( l \)-adic integer ring \( \mathbb{Z}_l \) over all rational primes \( l \). Define congruence subgroups of \( \text{GL}_2(\hat{\mathbb{Z}}) \) by

\[
V_1(p^a) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathbb{Z}}) \bigg| c \in p^a \mathbb{Z}, d - 1 \in p^a \mathbb{Z} \right\}.
\]

Regarding \( \hat{\mathbb{Z}} \) as a subring of \( F_f \), we consider \( V_1(p^a) \) to be an open-compact subgroup of \( \text{GL}_2(F_f) \). We consider modular forms \( f: \text{GL}_2(F_\Lambda) \to \mathbb{C} \) satisfying

\[
(0.1) \quad f(axu) = f(x) \det(u_\infty)^{-\dot{w}} j(u_\infty, z_0)^{-k} \quad \text{for all } a \in \text{GL}_2(F) \text{ and all } u = u_f u_\infty \text{ with } u_f \in V_1(p^a) \text{ and } u_\infty \in C_{\infty,+}.
\]

The space of cusp forms \( S_{k_0}^+(p^a; \mathbb{C}) \) we consider consists of functions on \( \text{GL}_2(F_\Lambda) \) satisfying, in addition to (0.1), the holomorphy condition at \( \infty \) and the cuspidal condition (for details, see §2). This space is naturally isomorphic to the
space of classical holomorphic Hilbert cusp forms of weight $k$ and of level $p^\alpha$. Then we can define Hecke operators $T_\mathfrak{n}(n)$ for each ideal $\mathfrak{n}$ acting on $S^*_k(u)(p^\alpha; \mathcal{C})$, and the Hecke algebra $\mathcal{H}_{k,u}(p^\alpha; \mathfrak{O}(v))$ is by definition the subalgebra of $\text{End}_\mathcal{C}(S^*_k(u)(p^\alpha; \mathcal{C}))$ generated over $\mathcal{O}(v)$ by $T_\mathfrak{n}(n)$ for all ideals $\mathfrak{n}$. The precise definition of $T_\mathfrak{n}(n)$ will be given in Section 3, which is a slight modification of the classical definition when $v \neq 0$. Let $K/Q_p$ be a finite extension inside $\overline{Q}_p$ and suppose that $K \supset \Phi(v)$. Then the $p$-adic integer ring $\mathcal{O}$ of $K$ contains $\mathcal{O}(v)$. We simply put, for $\mathfrak{A} = \mathcal{O}$ or $K$,

$$\mathcal{H}_{k,u}(p^\alpha; \mathfrak{A}) = \mathcal{H}_{k,u}(p^\alpha; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} \mathfrak{A}.$$ 

For each pair of integers $\beta > \alpha > 0$, it is well-known that there exists a surjective $\mathcal{O}$-algebra homomorphism $\rho^\beta: \mathcal{H}_{k,u}(p^\beta; \mathfrak{O}) \to \mathcal{H}_{k,u}(p^\alpha; \mathcal{O})$ which takes $T_\mathfrak{n}(n)$ to $T_\mathfrak{n}(n)$ for all $\mathfrak{n}$. Without having recourse to the theory of $p$-adic modular forms, we can define the $p$-adic Hecke algebra by

$$\mathcal{H}_{k,u}(p^\alpha; \mathfrak{O}) = \lim_{\alpha \to ^} \mathcal{H}_{k,u}(p^\alpha; \mathfrak{O}),$$

which, as will be seen in Section 4, acts naturally on the space of $p$-adic modular forms and is in fact the $\mathcal{O}$-linear dual space of the space of $p$-adic modular forms (Th. 5.3 in §5). The ordinary part $\mathcal{H}_{k,u}^{\text{ord}}(p^\alpha; \mathfrak{O})$ $(0 < \alpha \leq \infty)$ is the maximal algebra direct summand of $\mathcal{H}_{k,u}(p^\alpha; \mathfrak{O})$ on which the image of $T_\mathfrak{n}(p)$ is a unit. Then we have:

**Theorem I.** For any two weights $n$ and $n'$ with $n \sim n' \sim -2v$ and $n \geq n' \geq 0$, there exists an $\mathcal{O}$-algebra isomorphism:

$$\mathcal{H}_{k,u}^{\text{ord}}(p^\alpha; \mathfrak{O}) \cong \mathcal{H}_{k',u}^{\text{ord}}(p^\alpha; \mathfrak{O})$$

which takes $T_\mathfrak{n}(n)$ to $T_\mathfrak{n}(n)$, where $k' = n' + 2t$ and $w' = v + k' - t$.

By this theorem, the ordinary Hecke algebra $\mathcal{H}_{k,u}^{\text{ord}}(p^\alpha; \mathfrak{O})$ depends only on the class of $v \mod Z \cdot t$; so, we write $h_v^{\text{ord}}(1; \mathfrak{O})$ instead of $\mathcal{H}_{k,u}^{\text{ord}}(p^\alpha; \mathfrak{O})$. Similar independence of the whole Hecke algebra $\mathcal{H}_{k,u}(p^\alpha; \mathfrak{O})$ with respect to the weights will also be shown in Section 11 in the case where $v = 0$ by some results of Shimura [30] (see also Ohta [26]).

Let $Z$ be the Galois group of the maximal abelian extension $\tilde{\mathcal{F}}_{\infty}/F$ unramified outside $p$ and $\infty$. Let $\mathcal{F}_\alpha$ be the maximal ray class field modulo $p^\alpha$, and put

$$Z_\alpha = \text{Gal}(\tilde{\mathcal{F}}_{\infty}/\mathcal{F}_\alpha) \subset Z.$$ 

Let $Z_{\text{tor}}$ be the torsion part of $Z$ and decompose $Z = W \times Z_{\text{tor}}$ for a torsion free part $W$. Then $Z_\alpha$ and $W$ are $p$-profinite groups and for sufficiently large $\alpha$, $Z_\alpha \subset W$. Let $\mathbb{A}$, $\Lambda_\alpha$, and $\Lambda$ be the continuous group algebras over $\mathcal{O}$ of $Z$, $Z_\alpha$ and $W$, respectively. Then, $\Lambda$ is isomorphic to the formal power series ring over
of several variables (if the Leopoldt conjecture holds for \( F \), then \( \Lambda \cong \mathcal{O}[[X]] \) for one indeterminate \( X \)). Let \( \chi = \chi_l^i \colon \mathbb{Z} \to \mathbb{Z}_p^\times \) be the cyclotomic character, and for each \( l \in \mathbb{Z} \cdot t \), with \( l = [l]t \) for \([l] \in \mathbb{Z} \), we write \( \chi_l^i \) for \( \chi_l^{[l]} \colon \mathbb{Z} \to \mathbb{Z}_p^\times \).

Since \( \chi_l^i \colon \mathbb{Z} \to \mathbb{Z}_p^\times \) is a continuous character, we can extend it to an \( \mathcal{O} \)-algebra homomorphism \( \chi_{l,a}^i \colon \Lambda_a \to \mathcal{O} \). Let \( \omega_{l,a} = \operatorname{Ker}(\chi_{l,a}^i) \) which is a prime ideal of \( \Lambda_a \). Let \( \Lambda_{a,l} \) denote the localization of \( \Lambda_a \) at \( \omega_{l,a} \). Then we have:

**Theorem II.** There is a natural \( \mathcal{O} \)-algebra structure on \( \mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \) such that

\[
\mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \otimes_{\Lambda_a} \Lambda_{a,l}/\omega_{l,a} \Lambda_{a,l} \cong \mathcal{h}^{\text{ord}}_{k,w}(p^{\alpha}; \mathcal{O})
\]

for all \( n - 2v \) (\( n \geq 0 \)) and \( \alpha > 0 \), where \( l = n + 2v \), \( k = n + 2t \) and \( w = v + k - t \). Moreover \( \mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \) is a torsion-free \( \Lambda \)-module of finite type and is reduced.

Let \( \mathcal{S} \) be the quotient field of \( \Lambda \), and fix an algebraic closure \( \overline{Q} \) of \( \mathcal{S} \). To give an absolutely irreducible component of \( \operatorname{Spec}(\mathcal{h}^{\text{ord}}_v(1; \mathcal{O})) \) is equivalent to giving a \( \Lambda \)-algebra homomorphism \( \lambda \colon \mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \to \overline{Q} \). We fix such a \( \lambda \). Let \( \mathcal{X} \) be the quotient field of the image of \( \lambda \) and \( \mathfrak{S} \) be the integral closure of \( \lambda \) in \( \mathcal{X} \).

By extending the scalar field if necessary, we may assume that \( Q_p \cap \mathfrak{S} = \mathcal{O} \). We say that a \( Q_p \)-valued point \( P \) in \( \operatorname{Spec}(\mathfrak{S})/\mathcal{O} \) is algebraic if \( P \) is over \( \omega_{l,a} \in \operatorname{Spec}(\Lambda_a) \) for some \( \alpha > 0 \) and \( 0 \leq l \in \mathbb{Z} \cdot t \). We write this \( l \) as \( n(P) \), and the minimum of \( \alpha \) with the above property as \( \alpha(P) \). Regarding algebraic points \( P \) as an \( \mathcal{O} \)-algebra homomorphism: \( \mathfrak{S} \to Q_p \), we can define an \( \mathcal{O} \)-algebra homomorphism

\[
\lambda_P = P \circ \lambda \colon \mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \to Q_p.
\]

Then we have:

**Theorem III.** For each algebraic point \( P \) of \( \operatorname{Spec}(\mathfrak{S}) \) with \( n(P) \geq 2v \), \( \lambda_P(T_0(n)) \) is an algebraic number in \( \overline{Q} \) for all ideals \( n \), and there exists a non-trivial complex cusp form \( f_P \in S_{k,w}(p^{\alpha(P)}; \mathbb{C}) \) for \( k = n(P) - 2v + 2t \), \( w = n(P) - v + t \) such that \( f_P|T_0(n) = \lambda_P(T_0(n))f_P \) for all ideals \( n \). This cusp form \( f_P \) is determined up to constant multiple.

In fact, we can specify \( f_P \) by using the Fourier expansion of \( f_P \). Then, this correspondence: \( P \mapsto f_P \) can be extended to an algebraic function on \( \operatorname{Spec}(\mathfrak{S})(\overline{Q}_p) \) with values in the space of \( p \)-adic modular forms. This parametrization of common eigenforms is universal in the sense that for any given common eigenform \( f \in S_{k,w}(p^\alpha; \mathbb{C}) \) (\( k \geq 2t, k - 2v \)) whose eigenvalue for \( T_0(p) \) is a \( p \)-adic unit in \( Q_p \), we can find \( \lambda \colon \mathcal{h}^{\text{ord}}_v(1; \mathcal{O}) \to \overline{Q} \) from which \( f \) is obtained as in the theorem.
These theorems will be restated for arbitrary level in Section 3 and will be proved in Sections 11 and 12. We shall also prove the finiteness over \( \Lambda \) of the module of congruence and the module of differentials of \( \lambda \) in Section 3, which has at least conjecturally an intimate relation with the special values of \( L \)-functions of \( \text{GL}(3) \) as seen in [14] in the case of \( F = \mathbb{Q} \). Our method for proving these theorems relies firstly upon the analysis of the structure, as \( \Lambda \)-module, of cohomology groups of arithmetic subgroups of quaternion algebras over \( F \) and secondly upon the theory of \( p \)-adic Hilbert modular forms constructed by Deligne, Ribet, Rapoport and Katz. We shall give an exposition of the latter theory in Section 4 and will prove the duality between the Hecke algebras and the space of cusp forms in Section 5. We then analyse the above mentioned cohomology groups in the following Sections 6, 7, 8, 9 and 10 by adopting an idea of Shimura which goes back to 1960’s [30]. We shall generalize in these sections the results obtained in [14, §§3, 4 and 5] on cohomology groups coming from \( M_2(\mathbb{Q}) \) to those coming from quaternion algebras over \( F \) which are totally definite, or indefinite but yield Shimura curves. In the hope of having the same type of results for more general quaternion algebras yielding varieties of higher dimension, we have included some results which are irrelevant to our present purpose but are expected to be useful in the higher dimensional case.

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Notation. Throughout this paper, we fix a rational prime \( p \) and a totally real algebraic number field \( F \) of finite degree. We denote by \( \overline{\mathbb{Q}} \) the algebraic closure of \( \mathbb{Q} \) inside \( \mathbb{C} \). We also fix an algebraic closure \( \overline{\mathbb{Q}}_p \) of the \( p \)-adic field \( \mathbb{Q}_p \).
and an embedding: $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Thus any algebraic number in $\overline{\mathbb{Q}}$ can be considered uniquely as a $p$-adic number as well as a complex number. The normalized $p$-adic absolute value of $x \in \overline{\mathbb{Q}}_p$ will be denoted by $|x|_p$. We denote by $\Omega$ the completion of $\overline{\mathbb{Q}}_p$ under the norm $| \cdot |_p$.

By a quaternion algebra over $F$, we mean a central simple algebra over $F$ of dimension $4$; so, we include the matrix algebra $M_2(F)$ in this category. For each quaternion algebra $B/F$, we denote by $G = G^B$ the linear algebraic group defined over $\mathbb{Q}$ such that $G(\mathbb{Q}) = B^\times$. We denote by $G_\Lambda$ the adelization of $G$, and $G_f$ (resp. $G_\infty$, $G_{\infty+}$) denotes the finite part of $G_\Lambda$ (resp. the infinite part of $G_\Lambda$ and the connected component of $G_\infty$ with the identity). Similarly, for each finite extension $K/\mathbb{Q}$, we sometimes consider $K^\times$ (resp. $K$) as a non-split torus (resp. an additive group) defined over $\mathbb{Q}$ such that $K^\times(\mathbb{Q})$ (resp. $K(\mathbb{Q})$) is isomorphic to the multiplicative group (resp. the additive group) of $K$, and $K_\Lambda$, $K_\infty$, $K_f$ and $K_{\infty+}$ denote the adele ring of $K$, the infinite part of $K_\Lambda$, the finite part of $K_\Lambda$ and the connected component of $K_{\infty+}$ of the identity, respectively. We denote by $\nu: G \to F^\times$ the reduced norm map which can be viewed as a homomorphism of algebraic groups. For each place $\sigma$ of $F$, let $F_\sigma$ denote the completion of $F$ at $\sigma$. For each finite extension $K/\mathbb{Q}$, let $\mathfrak{z}_k$ denote the integer ring of $K$. We write simply $\mathfrak{z}$ for $\mathfrak{z}_F$. For each integral ideal $N$ of $\mathfrak{z}$, let $F_N = \prod_{\sigma|N} F_\sigma$. We denote by $x_N$ for $x \in G_\Lambda$, $F_\Lambda$ or $F_\Lambda^\times$ the projection of $x$ in $G(F_N)$, $F_N$ or $F_N^\times$. Especially, $x_\sigma$ (resp. $x_\infty$, $x_f$) denotes the $\sigma$-component (resp. the infinite part, the finite part) of $x$. We denote by $\mathfrak{z}_N$ (resp. $\mathfrak{z}_f$) the closure of $\mathfrak{z}$ in $F_N$ (resp. $F_f$). Then we know that $\mathfrak{z}_N = \prod_{\sigma|N} \mathfrak{z}_\sigma$. We put $\hat{\mathbb{Z}} = \prod_l \hat{\mathbb{Z}}_l$, where $l$ runs over all rational primes, and we put $\hat{\mathfrak{z}}_K = \mathfrak{z}_K \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. We can regard $\hat{\mathfrak{z}}_K$ as a subring of $K_f$. Each fractional ideal $\mathfrak{a}$ of $K$ can be expressed as $x\hat{\mathfrak{z}}_K \cap K$ (in $K_f$) for some $x \in K_f$. The ideal $x\hat{\mathfrak{z}}_K \cap K$ will be written as $x\mathfrak{z}_K$.

Let $I_K$ be the set of all embeddings of $K$ into $\overline{\mathbb{Q}}$. When $K = F$, we simply write $I$ for $I_F$. We denote by $A[I]$, for each commutative algebra $A$ and a subset $J$ of $I_K$, the $A$-free module generated by the elements of $J$. The module $A[I_K]$ has a natural right action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For $k = \sum_{\sigma \in I} k_\sigma \cdot \sigma \in \mathbb{R}[I]$, we write $k \geq 0$ if $k_\sigma \geq 0$ for all $\sigma \in I$, and $k > 0$ if $k \geq 0$ and $k \neq 0$. This positivity on $\mathbb{R}[I]$ is extended to an order on $\mathbb{R}[I]$ so that $k > k'$ (resp. $k \geq k'$) if $k - k' > 0$ (resp. $k - k' \geq 0$). We define a map $k: C^I \ni x \mapsto x^k \in C$ for each $k = \sum_{\sigma} k_\sigma \cdot \sigma \in \mathbb{Z}[I]$ by $x^k = \prod_{\sigma \in I} x_\sigma^{k_\sigma}$. When $x_\sigma > 0$ for all $\sigma \in I$, we can even define $x^s = \prod_{\sigma} x_\sigma^{s_\sigma}$ for $s \in \mathbb{C}[I]$. Since we can consider $F_{\infty}^\times$ as a subspace of $C^I$ naturally, the map $k$ induces a quasi-character: $F_{\infty}^\times \to C^\times$. We denote by $\Phi(k)$ the subfield of $\overline{\mathbb{Q}}$ generated by $x^k$ for all $x \in F$. Then $\Phi(k)$ is the fixed field of

$$\{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid k \sigma = k \}.$$
Let $e = e_F : F^\Lambda/F \to \mathbb{C}$ be the unique additive character such that

$$e_F(x) = \exp \left( 2\pi \sqrt{-1} \sum_{\sigma \in I} x_\sigma \right)$$

for $x = (x_\sigma) \in F^\infty$.

By abusing notation, we write for $x = (x_\sigma) \in \mathbb{C}^I, e_F(x) = \exp(2\pi \sqrt{-1} \sum_{\sigma \in I} x_\sigma)$; especially, for $\xi \in F$ and $x \in \mathbb{C}^I, e_F(\xi x) = \exp(2\pi \sqrt{-1} \sum_{\sigma \in I} \xi x_\sigma)$ is well defined.

For each $x \in F^\Lambda$, we write $|x|_\Lambda$ for the module of $x$ as in [38].

For each finite set $A$, $|A|$ denotes the cardinality of $A$. For any two sets $X$ and $Y$, we denote by $X^Y$ the set of all functions of $Y$ into $X$. There is one exception for this notation: If we are given a group $\Gamma$ and a $\Gamma$-module $X$, we put $X^\Gamma = H^0(\Gamma, X) = \{ x \in X | \gamma \cdot x = x \text{ for all } \gamma \in \Gamma \}$.

We trust there will be no confusion about this notation.

**1. Modules over quaternion algebras**

We take a quaternion algebra $B$ over $F$ and fix a maximal order $R$ of $B$. Let $\Sigma$ be the set of all places of $F$, and put

$$I_B = \{ \tau \in I | B \otimes_F F_\tau \cong M_2(F_\tau) \},$$

$$r = |I_B|, \quad \Sigma^B = \{ \tau \in \Sigma | B \otimes_F F_\tau \neq M_2(F_\tau) \}.$$

We consider $I$ as a subset of $\Sigma$ consisting of infinite places. Take a finite Galois extension $K_0/\mathbb{Q}$ (in $\mathbb{C}$) containing $F$, and denote by $\mathcal{O}_0$ its integer ring. Suppose that we have an isomorphism:

$$\tag{1.1} B \otimes_\mathbb{Q} K_0 \cong M_2(K_0)^I \text{ such that (i) the projection } \tau : B \to M_2(K_0) \text{ at each } \tau \in I_B \text{ takes } B \text{ into } M_2(K_0 \cap R), \text{ and (ii) } R \otimes_{\mathbb{Z}} \mathcal{O}_0 \text{ is sent into } M_2(\mathcal{O}_0)^I.$$

We can always find an extension $K_0$ and an isomorphism as in (1.1). For each $\mathcal{O}_0$-algebra $A$, we consider the polynomial ring $A[X, Y]$ with $2|I|$ indeterminates: $X = (X_\sigma)_{\sigma \in I}$ and $Y = (Y_\sigma)_{\sigma \in I}$. On each polynomial $P(X, Y)$ with coefficients in $A$, we get $\gamma = (\gamma_\sigma)_{\sigma \in I} \in M_2(\mathcal{O}_0)^I$ act via

$$P|_{\gamma}(X, Y) = \tilde{P}(X, Y)^I_{\gamma},$$

where $(X, Y)^I_{\gamma} = ((X_\sigma, Y_\sigma)^I_{\gamma_\sigma})_{\sigma \in I}$. Thus $A[X, Y]$ becomes a right module over the multiplicative semi-group $R$. Let $L(n; A)$ for $0 \leq n \in \mathbb{Z}[I]$ denote the $A$-submodule of $A[X, Y]$ consisting of all polynomials homogeneous for the variables $(X_\sigma, Y_\sigma)$ of degree $n_\sigma$ at every $\sigma \in I$. Then $L(n; A)$ is stable under the right action of $R$. For each $v \in \mathbb{Z}[I]$, we shall now twist the action of $R$ on
L(n; A) and define a new action by \( P_\gamma v = v(\gamma)^v P \vert \gamma \). The A-module \( L(n; A) \) equipped with this twisted right \( R \)-action will be denoted by \( L(n, v; A) \). More generally, for each A-module \( M \), we can consider the right \( R \)-module \( L(n, v; M) = L(n, v; A) \otimes_A M \) whose \( R \)-action is induced from \( L(n, v; A) \). This action can be naturally extended to a unique action of \( G^B(A) \). We can convert the right action of \( G^B(A) \) into a left action by 
\[
\gamma \cdot P = P_{\vert \gamma}^{-1}.
\]
This left \( G^B(A) \)-module will be denoted by \( L(n, v; M) \).

2. Spaces of cusp forms and Hecke operators

Put \( I^B = I - I_B = \Sigma^B \cap I \). For each \( k \in \mathbb{Z}[I] \), we define \( k^B \in \mathbb{Z}[I^B] \) and \( k_B \in \mathbb{Z}[I_B] \) by \( k^B = \sum_{\sigma \in I^b} k_\sigma \cdot \sigma \) and \( k_B = \sum_{\sigma \in I_B} k_\sigma \cdot \sigma \). We denote by \( t \in \mathbb{Z}[I] \) the special element \( t = \sum_{\sigma \in I} \sigma \). We firstly clarify what kind of conditions we shall impose on the weights of cusp forms. For two elements \( k, k' \in \mathbb{Z}[I] \), we write \( k \sim k' \) if \( k - k' \in \mathbb{Z} \cdot t \). We fix throughout the paper a class in \( \mathbb{Z}[I]/\mathbb{Z} \cdot t \) and choose a representative \( v \in \mathbb{Z}[I] \) of this class such that \( v \geq 0 \) and \( v_\sigma = 0 \) for some \( \sigma \in I \). Such an element \( v \) is uniquely determined. We take \( n \in \mathbb{Z}[I] \) satisfying \( n + 2v \sim 0 \) and \( n \geq 0 \) and put

\[
k = n + 2t, \quad w = v + k - t \quad \text{and} \quad \hat{w} = t - v = k - w.
\]

Then, we see

\[
(2.1) \quad k \sim 2w, \quad w \sim \hat{w} \sim -v \quad \text{and} \quad k \sim n.
\]

Under our choice of \( n \), the unit group

\[
E = \{ e \in \mathbf{q} \mid e^\sigma > 0 \text{ for all } \sigma \in I \} = R^\times \cap F_\infty^\times
\]

acts trivially on \( L(n, v; A) \) for any commutative algebra \( A \) over \( \mathbf{q}_0 \).

The isomorphism (1.1) induces an identification: \( G^B_\infty = \text{GL}_2(\mathbf{R})/O_2(\mathbf{R}) \times (\mathbf{H}^\times)^I \), where \( \mathbf{H} \) denotes the Hamilton quaternion algebra over \( \mathbf{R} \). Let \( H \) be the upper half complex plane, and put \( \mathcal{D} = \mathcal{D}_B = H^I \). We can identify \( H \) with \( \text{GL}_2(\mathbf{R})/O_2(\mathbf{R}) \cdot \mathbf{R}^\times \) by

\[
\text{GL}_2(\mathbf{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a\sqrt{-1} + b)/(c\sqrt{-1} + d) \in H \quad \text{if } ad - bd > 0 \quad \text{and}
\]

\[
\text{GL}_2(\mathbf{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (-a\sqrt{-1} + b)/(-c\sqrt{-1} + d) \in H \quad \text{if } ad - bc < 0.
\]

Thus \( G_\infty \) naturally acts on \( z \) by \( g(z) = (\alpha_\sigma(z_\sigma))_{\sigma \in I} \). Especially, \( G_{\infty +} \) acts on \( \mathcal{D} \) complex analytically. Put \( z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathcal{D} \) and

\[
C_\infty = \{ g \in G_\infty \mid g(z_0) = z_0 \} \quad \text{and} \quad C_{\infty +} = C_\infty \cap G_\infty^+. 
\]
Then, we can identify
\[ C_\infty = (\mathbb{R}^\times O_2(\mathbb{R}))^{I_B} \times (\mathbb{H}^\times)^{I_B}, \quad C_{\infty +} = (\mathbb{R}^\times SO_2(\mathbb{R}))^{I_B} \times (\mathbb{H}^\times)^{I_B}. \]

Especially we know that \( C_\infty / C_{\infty +} \cong (\mathbb{Z}/2\mathbb{Z})^{I_B} \) as groups. For each subset \( J \) of \( I_B \) and \( x \in G_\infty \), we define another subset \( J^x \subset I_B \) by
\[ J^x = \{ \tau \in I_B | \tau \in J \text{ and } \nu(x_\tau) > 0, \text{ or } \tau \in \tilde{I} = I_B - J \text{ and } \nu(x_\tau) < 0 \}. \]

Then one verifies that this gives an action of \( G_\infty \) on subsets of \( I_B \); i.e.,
\[ J^{xy} = (J^x)^y \quad \text{for } x, y \in G_\infty. \]

For each subset \( J \) of \( I_B \), we shall define an automorphic factor \( j_J(x, z) \in C^{I_B} \) for \( z \in \mathcal{D}_B \) and \( x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) \( \in G_\infty \) by
\[ j_J(x, z) = (c_\sigma z_\sigma^J + d_\sigma)_{\sigma \in I_B}, \]
where
\[ z_\sigma^J = \begin{cases} z_\sigma & \text{if } \sigma \in J, \\ \tilde{z}_\sigma & \text{if } \sigma \in \tilde{I} = I_B - J. \end{cases} \]

Then, by definition, we have
\[ j_J(\gamma, z) = j_{I_B}(\gamma, z^J), \]
and as an element of the semi-simple algebra \( C^{I_B} \), \( j_J(\gamma, z) \) satisfies
\[ j_J(\gamma \delta, z) = j_J(\gamma, \delta(z)) j_J(\delta, z) \quad \text{for } \gamma, \delta \in G_\infty. \]

For each function \( f: G^B_A \to L(n_B, v_B; \mathbb{C}) \), we define a transform \( f|_{k, w, J} u \) of \( f \) under \( u \in G^B_A \) by
\[ (f|_{k, w, J} u)(x) = j_{\nu}(u_\infty, z_0)^{-k_B} \nu(u_\infty)^{w_B} f(xu^{-1}) \cdot u_\infty, \]
where \( u_\infty \) acts from the right on the value \( f(xu^{-1}) \) in \( L(n_B, v_B; \mathbb{C}) \). By (2.2a, b, c), we have the compatibility relation:
\[ (f|_{k, w, J} x)|_{k, w, J} y = f|_{k, w, J}(xy). \]

When it is unlikely to cause misunderstanding, we simply write \( f|x \) or \( f|_{k, w} x \) for the transform \( f|_{k, w, J} x \). Let \( U \) be an open-compact subgroup of \( G^B_f \). We now denote by \( S_{k, w, J}(U; B; \mathbb{C}) = S_{k, w, J}(U; \mathbb{C}) \) the space of functions \( f: G^B_A \to L(n_B, v_B; \mathbb{C}) \) satisfying the following conditions (2.4a, b, c, d):
\[ f = f|_{k, w, J} u \quad \text{for all } u \in UC_{\infty +}, \quad \text{and} \]
\[ f(ax) = f(x) \quad \text{for all } a \in G^B(B(\mathbb{Q}). \]
For each \( z \in \mathcal{F}_B \), we can choose \( u_\infty \in G_{\infty}^+ \) so that \( u_\infty(z_0) = z \) and define a function \( f_x: \mathcal{F}_B \to \mathcal{L}(n^B, v^B; C) \) for each \( x \in G_f^B \) out of each function \( f \) satisfying (2.4a) by

\[
f_x(z) = j_z(u_\infty, z_0)^k \nu(u_\infty)^{-w_x} f(xu_\infty) \cdot u_\infty^{-1}.
\]

Then \( f_x \) is well defined independently of the choice of \( u_\infty \) by (2.4a) as a function on \( \mathcal{F}_B \) with values in \( \mathcal{L}(n^B, v^B; C) \). Then we impose

\[
\text{For all } x \in G_f^B, \quad \frac{\partial f_x}{\partial z_\sigma} = 0 \quad \text{if} \quad \sigma \in J, \quad \text{and}
\]

\[(2.4b) \quad \frac{\partial f_x}{\partial z_\sigma} = 0 \quad \text{if} \quad \sigma \in \tilde{J} = I_B - J.
\]

In the extreme cases: \( r = 0 \) (i.e., \( B \) is totally definite) or \( B = M_2(F) \), we need to impose some other conditions:

\[(2.4c) \quad \text{When } B = M_2(F), \text{ then } \int_{F_{A/F}} f \left( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x \right) \, \text{d}a = 0 \quad \text{for all } x \in G^B_A.
\]

(When \( B = M_2(Q) \), we also add the following condition: \( |\text{Im}(z)^{k/2} f_x(z)| \) is uniformly bounded on \( H \) for all \( x \in GL_2(Q_f) \)). When \( B \) is totally definite and \( n = 0 \) (then \( v = 0 \)), we firstly consider the space \( S(U) \) of all functions on \( G^B_A \) satisfying (2.4a). Let \( \text{Inv}(U) \) be the subspace of \( S(U) \) consisting of functions of the form \( f \circ \nu \) for some function \( f: F^\times_A \to C \), where \( \nu: G^B_A \to F^\times_A \) is the reduced norm map. Then we put

\[(2.4d) \quad S_{2t, A, \phi}(U; B; C) = S(U) / \text{Inv}(U) \quad \text{if } n = 0 \quad \text{and} \quad r = 0.
\]

We have now finished defining the space of cusp forms for each \( B/F \). In our definition, we have (implicitly) assumed that \( k \geq 2t \) since \( n \geq 0 \). It is well-known that the space \( S_{k, A, j}(U; C) \) is of finite dimension.

By the approximation theorem, we can find \( t_i \in G_A \) (\( i = 1, \ldots, h \)) with \( (t_i)_\infty = (t_i)_N = 1 \) for any given ideal \( N \) of \( \mathfrak{p} \) such that \( G^B_A = \prod_{i=1}^h G_{Q_i}^1 t_i U G_{\infty}^+ \). When \( B \) is indefinite, the number \( h \) is equal to \( |F^\times \setminus F^\times_A / \nu(U) F^\times_{\infty} | \) and is independent of \( B \) by the strong approximation theorem. When \( B \) is (totally) definite, \( h \) depends on \( U \) and \( B \). We put, by fixing such a decomposition,

\[
\Gamma^i(U) = G^B_Q \cap t_i^1 U G^B_{\infty}, \quad \Gamma^i(U) = \Gamma^i(U) / \Gamma^i(U) \cap F^\times.
\]

Then \( \Gamma^i(U) \) is a discrete arithmetic subgroup of \( G_{\infty}^+ \). We then consider the space of cusp forms with respect to \( \Gamma^i(U) \), which is written as \( S_{k, j}(\Gamma^i(U); C) \)
and consists of functions \( f: \mathcal{Z}_B \to 'L(n^B, v^B; C) \) satisfying the following conditions:

\[(2.5a) \quad f(\gamma(z)) = \nu(\gamma)^{-w_B} j_{1}^{'}(\gamma, z)^{k}(\gamma \cdot f(z)) \quad \text{for all } \gamma \in \Gamma^i(U), \]

\[(2.5b) \quad \frac{\partial f}{\partial z_\sigma} = 0 \quad \text{if } \sigma \in I, \quad \text{and} \quad \frac{\partial f}{\partial z_\sigma} = 0 \quad \text{if } \sigma \in \tilde{J}. \]

When \( B = M_2(F) \), we suppose the cuspidal condition:

\[(2.5c) \quad f \text{ vanishes at all cusps of } \Gamma^i(U). \]

(This condition means that for all \( \alpha \in \text{SL}_2(F) \), \( f |_\alpha(z) = \nu(\alpha)^{w_B} j_{1}^{'}(\alpha, z)^{k} f(\alpha(z)) \)
has Fourier expansion of the form \( \sum_{\xi \in L} a(\xi) e_{\mathfrak{f}}(\xi z) (z \in \mathcal{Z} = \mathcal{Z}_1) \) for a lattice \( L \) depending on \( \alpha \), where \( L_+ \) is the subset of all totally positive elements of \( L \).

Then the correspondence: \( f \mapsto (f_{i}^{j})_{i} \) gives an isomorphism:

\[(2.6a) \quad S_{k, w, j}(U; C) \cong \bigoplus_{i=1}^{h} S_{k, w, j}(\Gamma^i(U); C) \quad \text{if } k > 2t \quad (n > 0) \text{ or } B \text{ is indefinite}. \]

Note that if \( B \) is totally definite, we have the trivial identity

\[(2.6b) \quad H^0(\Gamma^i(U), 'L(n, v; C)) = S_{k, w, j}(\Gamma^i(U); C) \quad \text{if } k > 2t. \]

The assertion (2.6a) follows from the following formula (2.6c): We define for \( f: \mathcal{Z}_B \to 'L(n^B, v^B, C) \) another function \( f|_{k, w, j}: \mathcal{Z}_B \to 'L(n^B, v^B, C) \) for \( \gamma \in G^B_Q \) by

\[(f|_{k, w, j})(\gamma)(z) = \nu(\gamma)^{w_B} j_{1}^{'}(\gamma, z)^{-k}(\gamma^{-1} \cdot f(\gamma(z))). \]

Put, for each \( x \in G^F C_\infty, U^x = x^{-1} U x \) and decompose

\[G^B_A = \prod_{i} G^B_Q |_{U^x} G^B_{\infty} \]

(for example, the choice \( t_i' = t_i x \) works well). Then, if \( t_j x \in \gamma t_i' U^x C_\infty \) for \( \gamma \in G^B_Q \), then

\[(2.6c) \quad (f|_{k, w, j})(t_i')_{i'} = (f_{i'}^{j})_{t_i' \cdot t_i}. \]

For the proof of (2.6c), see for example [29, §3] and [8, §1].

We shall now define the Hecke operators on \( S_{k, w, j}(U; C) \). Let \( U \) and \( U' \) be two open compact subgroups of \( G_F \). For each \( x \in G^F C_\infty \), we shall define a
linear operator $[UxU']$: $S_{k,w,f}(U; C) \rightarrow S_{k,w,f}(U'; C)$ as follows: Decompose $(UC_\infty+)x(UC_\infty+) = \bigcup_i(UC_\infty+)x_i$ as a disjoint union of finitely many left cosets. Then we define

$$f|[UxU'] = \sum_i f|_{k,w,f}x_i.$$ 

By (2.4a), this operator is independent of the choice of the representative set $\{x_i\}$ of the left cosets. By (2.2a, c) and (2.6c), the operator $[UxU']$ takes $S_{k,w,f}(U; C)$ into $S_{k,w,f}(U'; C)$. Note that $Ux_fU' = \bigcup_i(Ux_i)_f$ if and only if $(UC_\infty+)x(UC_\infty+) = \bigcup_i(Ux_i)x_\infty$. Because of this fact, we have used the symbol $[UxU']$ to denote this operator instead of $[UC_\infty+,xUC_\infty+]$. Through the operator $[UxU']$ for $x \in C_\infty\subset N_C$, the finite group $C_\infty/C_\infty$ acts on the sum $\bigoplus_{i \in \mathbb{Z}} S_{k,w,f}(U; C)$. In the extreme case of $r = 0$ and $n = 0$, the transformation: $f \mapsto f|_x$ preserves by definition the subspace $\text{Inv}(U)$. Thus the operator $[UxU']$ induces a linear operator:

$$S_{2t,\phi}(U; C) \rightarrow S_{2t,\phi}(U'; C),$$

which is again denoted by $[UxU']$.

If $U' \subset U$, $S_{k,w,f}(U; C)$ is naturally contained in $S_{k,w,f}(U'; C)$. Thus we can take the injective limit:

$$(2.7) \quad S_{k,w,f}(B; C) = \lim_{\longrightarrow U} S_{k,w,f}(U; B; C)$$

over the partially ordered set of all open compact subgroups of $G_f$. By (2.3b), for each $f \in S_{k,w,f}(B; C)$, the transform $f|_{k,w,f}x$ is again contained in $S_{k,w,f}(B; C)$ for $x \in G_f$. Thus $G_f$ naturally acts on $S_{k,w,f}(B; C)$. When $B$ is unramified at every finite place (then $r \equiv [F:Q] \mod 2$), we shall fix an isomorphism of $\hat{R} = R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ with $M_2(\hat{\mathbb{Z}})$. Then every open compact subgroup $U$ of $GL_2(F_f)$ can be regarded as an open compact subgroup of $G^B_f$. Then, by virtue of a result of Jacquet, Langlands and Shimizu, we have:

**Theorem 2.1.** Let $B$ and $B'$ be quaternion algebras over $F$ unramified at all finite places. Suppose that $n \geq 0$ (i.e. $k \geq 2t$) and $k + 2v \sim 0$. Then for open compact subgroups $U$ of $GL_2(F_f)$, there is a system of isomorphisms

$$i_U: S_{k,w,f}(U; B; C) \cong S_{k,w,f}(U; B'; C)$$
such that (i) for $U \supset U'$, there is a commutative diagram:

$$
\begin{array}{ccc}
S_{k, w, j_B}(U; B; C) & \cong & S_{k, w, j_B}(U; B'; C) \\
\downarrow & & \downarrow \\
S_{k, w, j_B}(U'; B; C) & \cong & S_{k, w, j_B}(U'; B'; C),
\end{array}
$$

and (ii) if $i = \lim_U i_U$:

$$
S_{k, w, j_B}(U; B; C) \cong S_{k, w, j_B}(B'; C),
$$

then $i$ is an isomorphism of $GL_2(F_f)$-module; in particular,

$$
i_U \circ [UxU'] = [UxU'] \circ i_U
$$

for every pair of open compact subgroup $(U, U')$ of $GL_2(F_f)$ and every $x \in GL_2(F_f)$.

This follows from a result in [18, §16]. An exposition can be found in [8, §2].

As we have already remarked, for each subset $J$ of $I_B$, we can find $c = c(J) \in C_{\infty}$ such that $I_B = J^c(J)$. Since $c_f = 1$, the operator $[UxU']$ commutes with $[UxU]$ for all $x \in G_f^B$. Note that $[UxU]^2 = id$, and thus, $[UxU]$ is actually an isomorphism. Namely, we have:

**Theorem 2.2.** For each subset $J$ of $I_B$, the map

$$[UxU'] : S_{k, w, j_B}(U; C) \rightarrow S_{k, w, j_B}(U; C)
$$

is a surjective isomorphism satisfying

$$[UxU'] \circ [UxU] = [UxU] \circ [UxU] \quad \text{for all } x \in G_f^B.
$$

Let $i : G_A \rightarrow G_A$ denote the involution defined by $xx' = \nu(x)$. Note that we have defined $\hat{\nu} = t - v = k - w$. Thus $k \sim 2w$ and we may consider the space $S_{k, \hat{\nu}, j}(U; C)$. Put $U' = \{ x' \mid x \in U \}$. For each $f \in S_{k, \hat{\nu}, j}(U; C)$, we define

$$f^* : G_A \rightarrow L(n^B, -n^B - v^B; C) \quad \text{by } f^*(x) = f(x^{-1}),
$$

where we have written $x^{-1}$ for $(x^{-1})' = (x')^{-1}$.

**Proposition 2.3.** The correspondence: $f \mapsto f^*$ induces an isomorphism:

$$S_{k, \hat{\nu}, j}(U; C) \cong S_{k, \hat{\nu}, j}(U'; C)
$$

which satisfies $(f)[UxU]^* = f^*[[UxU']]$ for all $x \in UC_{\infty}$.

**Proof.** The $U'$-invariance of $f^*$ as in (2.4a) follows from a direct calculation. The cuspidal condition for $f^*$ in (2.4c) is obvious since $(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix})^t = (\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix})$. We shall show the analyticity of $f^*$ at infinite places. A straightforward computation
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shows that \( (f^*)_x(z) = f_x(z) \). This shows the expected analyticity. The last assertion follows simply from the definition of the operator \([U x U] \).

For each integral ideal \( N \) of \( F \) disjoint from \( \Sigma^B \), we may suppose that the isomorphism (1.1) induces another isomorphism

\[
R_N = R \otimes \mathbb{Z} N \cong M_2(\mathbb{Z} N).
\]

Then we shall define standard open compact subgroups of \( \hat{R}^\times \) (for \( \hat{R} = R \otimes \mathbb{Z} \hat{\mathbb{I}} \)) by

\[
U_0(N) = \left\{ x \in \hat{R}^\times \mid x_N = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid c \in N_2 \right\},
\]

\[
U_1(N) = \left\{ x \in U_0(N) \mid x_N = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a - 1 \in N_2 \right\},
\]

\[
V_1(N) = \left\{ x \in U_0(N) \mid x_N = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid d - 1 \in N_2 \right\},
\]

\[
U_E(N) = E \cdot U_1(N) \quad \text{and} \quad V_E(N) = E \cdot V_1(N),
\]

where \( E = \{ e \in \mathbb{Z}^\times \mid e^\sigma > 0 \text{ for all } \sigma \in I \} \) and the product of \( E \) with \( U_1(N) \) and \( V_1(N) \) is taken in \( G_f^B \). Suppose that \( f \in S_k, w, j(U_1(N); C) \) or \( f \in S_k, w, j(V_1(N); C) \). Then we see from (2.4a) that

\[
f(x e_c) = e_{2w-k} f(x) \quad \text{for } c \in F_{-1}^x.
\]

We know from the fact: \( 2w - k = 1 \) for all \( e \in E \). Therefore

\[
f(x) = f(x e f) = f(x e f e_c) = f(x e f) \quad \text{for } e \in E.
\]

This shows that

\[
S_{k, w, j}(U_1(N); C) = S_{k, w, j}(U_E(N); C),
\]

\[
S_{k, w, j}(V_1(N); C) = S_{k, w, j}(V_E(N); C).
\]

We simply write \( S_{k, w, j}(N; B; C) \) for \( S_{k, w, j}(V_1(N); B; C) \) and \( S_{k, w, j}(N; B; C) \) for \( S_{k, w, j}(U_1(N); B; C) \). Now we shall define Hecke operators \( T(n) \) and \( T(n, n) \) on these spaces. Put, for \( \hat{R} = R \otimes \mathbb{Z} \hat{\mathbb{I}} \),

\[
\Delta_1(N) = \left\{ x \in \hat{R} \cap G_f^B \mid x_N = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a - 1 \in N_2, c \in N_2 \right\}
\]

and

\[
\Delta_2(N) = E \cdot \Delta_1(N).
\]

Then these are semi-groups containing \( U_1(N) \) or \( U_E(N) \), and we can form the abstract Hecke ring \( R(U_1(N), \Delta_1(N)) \) and \( R(U_E(N), \Delta_2(N)) \) as in [36, III] (see also [7, §2]). Then the following facts can be verified in exactly the same manner.
as in [36] and [7]:

\[(2.9a) \quad R(U_1(N), \Delta_1(N)) \cong R(U_E(N), \Delta_E(N)) \cong R(V_E(N), \Delta_E(N)^{-1}) \cong R(V_1(N), \Delta_1(N)^{-1})\]

\[U_1(N)xU_1(N) \to U_E(N)xU_E(N) \to V_E(N)x^{-1}V_E(N) \to V_E(N)x^{-1}V_E(N),\]

\[(2.9b) \quad \text{For each ideal } M \subset \mathcal{O} \text{ disjoint from } \Sigma^B, \text{ we have an isomorphism of rings:} \]

\[R(U_1(NM^\alpha), \Delta_1(NM^\alpha)) \cong R(U_1(NM^\beta), \Delta_1(NM^\beta)) \quad \text{for all } \alpha \geq \beta > 0\]

\[U_1(NM^\alpha)xU_1(NM^\alpha) \to U_1(NM^\beta)xU_1(NM^\beta).\]

For each ideal \( m \subset \mathcal{O} \), put \( \mathcal{I}(m) = \{ x \in \Delta_1(N) | \nu(x) = m \} \). We decompose \( \mathcal{I}(m) \) into a disjoint union \( \bigsqcup_j U_1(N)xU_1(N) \) of finitely many double cosets, which is always possible, and we use the same symbol \( \mathcal{I}(m) \) for the element \( \sum_j U_1(N)xU_1(N) \) in \( R(U_1(N), \Delta_1(N)) \). By choosing an element \( m \in \mathfrak{f} \cap \mathcal{O} \) such that \( m = m_1 \) and \( m - 1 \in N \mathfrak{f} \) (then \( m \in \Delta_1(N) \)) if \( n \) is prime to \( N \) and disjoint from \( \Sigma^B \), we put

\[\mathcal{I}(m, m) = \begin{cases} U_1(N)mU_1(N) & \text{if } m \text{ is prime to } N \text{ and is disjoint from } \Sigma^B, \\ 0 & \text{otherwise}. \end{cases}\]

Then \( \mathcal{I}(m, m) \) is determined independently of the choice of \( m \), and we have

\[(2.9c) \quad R(U_1(N), \Delta_1(N)) \text{ is isomorphic to the polynomial ring over } \mathbb{Z} \text{ of variables } \mathcal{I}(\ell, \ell) (\ell \nmid N, \ell \notin \Sigma^B) \text{ and } \mathcal{I}(\ell) \text{ for all prime ideals } \ell,\]

\[(2.9d) \quad \mathcal{I}(\ell)^2 - \mathcal{I}(\ell^2) = \mathcal{N}_E/Q(\ell)\mathcal{I}(\ell, \ell) \text{ for each prime ideal } \ell \text{ outside } N \text{ and } \Sigma^B.\]

Now we identify all the Hecke rings by the isomorphism (2.9a) and write it simply as \( \mathcal{R} = \mathcal{R}(N) \). Then

\[(2.9e) \quad \mathcal{R}(N) \text{ acts on } S_{k,w}^+(N; B; C) \text{ via } U_1(N)xU_1(N) \to [U_1(N)xU_1(N)] \text{ and on } S_{k,w}^+(N; B; C) \text{ via } U_1(N)xU_1(N) \to [V_1(N)x^{-1}V_1(N)].\]

The operator corresponding to \( \mathcal{I}(n) \), \( \mathcal{I}(n, n) \) on these spaces will be denoted by \( \mathcal{I}(n) \) and \( \mathcal{I}(n, n) \). Then by Proposition 2.3, we have a commutative
diagram
\[
S_{k,w,J}(N;C) \equiv S_{k,w,J}(N;\mathbb{C})
\]
(2.10)
\[
\begin{array}{c}
T(\alpha), T(\alpha, \eta) \\
S_{k,w,J}(N;C) \equiv S_{k,w,J}(N;C).
\end{array}
\]

By our definition of the transformation: \( f \rightarrow f|_x \) in (2.3a), the action of \( T(\alpha) \) and \( T(\alpha, \eta) \) on \( S_{k,w,J}(N;\mathbb{C}) \) given here coincides with the Hecke operators on the space of holomorphic cusp forms defined by Shimura [29], [34] and Weil [39] for a suitable choice of \( w \) for each \( k \). For example, in [34, §2], \( k/2 \) is taken as \( w \). This choice is not appropriate, when \( k \neq 0 \) mod \( 2\mathbb{Z}[I] \), for the analysis of integrality of Hecke operators which will be done in the coming sections. The use of integral \( w \) was initiated by Shimura in [29] and [30].

3. Results on Hecke algebras

In order to define the Hecke algebra for the space of cusp forms, we shall modify a little the Hecke operators \( T(\alpha) \) and \( T(\alpha, \eta) \) when \( v \neq 0 \). The case: \( v = 0 \) corresponds to the classical parallel weight cusp forms and in this case, no modification is necessary. For each \( \xi \in \mathbb{Z}[I] \), let
\[
\mathcal{G}(\xi) = \{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})|\xi\sigma = \xi \}.
\]
Then it is easy to see that \( \mathcal{G}(\xi) = \mathcal{G}(\eta) \) if \( \xi \sim \pm \eta \), since \( \mathcal{G}(t) \) is the whole group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Let \( \Phi(\xi) = \{ x \in \overline{\mathbb{Q}}|x^\sigma = x \text{ for all } \sigma \in \mathcal{G}(\xi) \} \). We then consider a quasi-character \( \xi: F^\times \rightarrow \overline{\mathbb{Q}}^\times \) given by \( x \rightarrow x^\xi = \prod_{\sigma \in \mathcal{G}(\xi)}x^{\xi\sigma}. \) Then \( \Phi(\xi) \) is the subfield of \( \overline{\mathbb{Q}} \) generated by \( x^\xi \) for all \( x \in F^\times \), and thus the quasi-character \( \xi \) has values in the finite extension \( \Phi(\xi) \). Therefore the character \( \xi \) extends by continuity to characters: \( F^\times \rightarrow \Phi(\xi)^\times, F^\times_p \rightarrow \Phi_p^\times \) and \( F^\times_\infty \rightarrow C^\times \). We write simply \( \mathfrak{z}(v) \) for \( \mathfrak{z}(\Phi(\xi)) \). Let \( A \) be an \( \mathfrak{z}(v) \)-algebra inside \( C \), and suppose
\[
(3.1) \quad \text{For every } x \in F^\times, \text{ the } A\text{-ideal } x^vA = (x^v \mathfrak{z}(v))A \text{ is generated by a single element in } A.
\]
We can find a finite extension \( K_0/\Phi(v) \) such that the integer ring \( \mathfrak{z}_0 \) of \( K_0 \) satisfies (3.1). In fact, by choosing elements \( \{ a_i \}_{i=1,\ldots,h} \) of \( F^\times_f \) such that \( a_i^\mathfrak{z} \) gives a complete representative set for all ideal classes of \( F \), we can take \( a_i \in \mathfrak{z} \) so that \( a_i^\mathfrak{z} = (a_i^\mathfrak{z})^h \). Then, as an example of such an extension, we can take the field generated over \( \Phi(v) \) by \( \sqrt{\alpha_i^v} \) for all \( i = 1,\ldots,h \). As another example of \( A \) satisfying (3.1), we may of course take the valuation ring of \( \Phi(v) \) at each finite place. When \( v = 0 \) (i.e. \( w \sim 0 \)), the rational integer ring \( \mathbb{Z} \) satisfies the condition (3.1), and thus the condition (3.1) imposes no restriction to the ring \( A \).
For each prime ideal ℓ of r, we take x ∈ F'_ℓ such that ℓ = xℓ and choose, once and for all, a generator \{x^v\} = \{ℓ^v\} of x^vA. For a general ideal a of F, we decompose a = \prod_ℓ a^v(ℓ) as a product of prime ideals and put

\{a^v\} = \prod_ℓ \{ℓ^v\}^{a^v(ℓ)},

which gives a generator of the ideal x^vA for x ∈ F'_ℓ with xℓ = a. We also write \{x^v\} for \{a^v\} if a = xℓ for x ∈ F'_ℓ. The correspondence: a → \{a^v\} gives a multiplicative map of the ideal group of F into the quotient field of A but is not necessarily a Hecke character. We shall define operators

\((U_1(N) x U_1(N))': S_{k, w, f}(N; B; C) \rightarrow S_{k, w, f}(N; B; C)\) by

\[f|[(U_1(N) x U_1(N))] = \{\nu(x)^v\}^{-1} f|[U_1(N) x U_1(N)]\quad \text{for } x \in \Delta_1(N).\]

Similarly, we define \((U_1(N) x U_1(N)) \in \text{End}_C(S_{k, w, f}(N; B; C))\) by

\[f|[U_1(N) x U_1(N))] = \{\nu(x)^v\}^{-1} f|[V_1(N) x^{-1} V_1(N)]\quad \text{for } x \in \Delta_1(N).\]

By decomposing \(\mathcal{T}(n) = \prod_i U_1(N) x_i U_1(N)\) (then, ν(x_i) = n for all i), we put

\[T_0(n) = \sum_i (U_1(N) x_i U_1(N)) = \{n^v\}^{-1} T(n), \quad T_0(n, n) = \{n^{2v}\}^{-1} T(n, n)\]

as operators on \(S_{k, w, f}(N; C)\) and \(S_{k, w, f}^*(N; C)\). Hereafter in this section, we suppose

\[\text{(3.2) } B \text{ is unramified at all finite places (i.e. } \Sigma B \subset I),\]

and we shall identify \(\hat{R} = R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\) with \(M_2(\hat{\mathbb{Z}})\). Then the Hecke algebra \(\mathcal{H}_{k, w}(N; A)\) is by definition the A-subalgebra of \(\text{End}_C(S_{k, w, f}(N; B; C))\) generated over A by the operators \(T_0(n)\) for all integral ideals n. By Theorems 2.1 and 2.2, \(\mathcal{H}_{k, w}(N; A)\) is determined independently of the choice of the quaternion algebra B under (3.2) and the subset \(J\) of \(I_B\). It is also plain that the A-algebra \(\mathcal{H}_{k, w}(N; A)\) is independent of the choice of the map: a → \{a^v\}. By (2.9d), we have the relation:

\[T_0(\ell)^2 - T_0(\ell^2) = \mathcal{N}_{F/Q}(\ell) T_0(\ell, \ell) \quad \text{for prime ideals } \ell \nmid N.\]

Thus if \(\mathcal{N}_{F/Q}(\ell)\) is invertible in A, \(T_0(\ell, \ell)\) is contained in \(\mathcal{H}_{k, w}(N; A)\) (this statement is actually true without the assumption that \(\mathcal{N}_{F/Q}(\ell)^{-1} \in A\) as will be seen later in this section). The following fact may be well-known, but we will give a proof in Section 7:
Theorem 3.1. In addition to (3.1), suppose one of the following conditions for A:

(i) A is the integer ring of a finite extension of $\Phi(v)$,
(ii) A is a discrete valuation ring of $\Phi(v)$,
(iii) A is a field extension of $\Phi(v)$.

Then

(3.3a) For any $A$-algebra $D$ in C, $\mathcal{H}_{k, u}(N; D) \cong \mathcal{H}_{k, u}(N; A) \otimes_A D$,

(3.3b) $\mathcal{H}_{k, u}(N; A)$ is a flat $A$-module of finite type.

Let $\mathcal{O}(v)$ be the valuation ring of $\Phi(v)$ corresponding to the fixed embedding: $\Phi(v) \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. For any $\mathcal{O}(v)$-algebra $A$ not necessarily inside C, we put

\[
\mathcal{H}_{k, u}(N; A) = \mathcal{H}_{k, u}(N; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} A.
\]

By (3.3a), this definition is compatible with the base change of the ring A. Let $\hat{\mathcal{O}}(v)$ be the $p$-adic closure of $\mathcal{O}(v)$ in $\overline{\mathbb{Q}}_p$. Fix a valuation ring $\mathfrak{O}$ containing $\hat{\mathcal{O}}(v)$ which is finite flat over $\mathbb{Z}_p$. We now fix an integral ideal $N$ prime to $p$. Then for $\alpha \geq \beta > 0$, we have a commutative diagram (cf. [36, III], [7, §2]):

\[
\begin{array}{ccc}
S_{k, u, f}(Np^\beta; C) & \longrightarrow & S_{k, u, f}(Np^\alpha; C) \\
\downarrow T_0(\mathfrak{n}) & & \downarrow T_0(\mathfrak{n}) \\
S_{k, u, f}(Np^\beta; C) & \longrightarrow & S_{k, u, f}(Np^\alpha; C)
\end{array}
\]

for all ideals $\mathfrak{n}$. Thus the restriction of operators in $\mathcal{H}_{k, u}(Np^\alpha; A)$ to the subspace $S_{k, u, f}(Np^\beta; C)$ induces a surjective $A$-algebra homomorphism:

(3.6a) $\mathcal{H}_{k, u}(Np^\alpha; A) \rightarrow \mathcal{H}_{k, u}(Np^\beta; A)$ for $\alpha \geq \beta > 0$,

which takes $T_0(\mathfrak{n})$ to $T_0(\mathfrak{n})$. By (3.3b), $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$ is a $p$-adically complete semi-local ring and hence is a product of finitely many local rings. Let $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$ be the product of all local factors of $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$ on which the projected image of $T_0(p)$ is a unit. The change of the map: $a \mapsto \{ a^\alpha \}$ affects $T_0(p)$ by multiplication of an $\mathfrak{O}$-unit; hence, $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$ is well-defined independently of the choice of the map: $a \mapsto \{ a^\alpha \}$. Let $e = e_\alpha$ be the idempotent of $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$ in the Hecke algebra $\mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O})$. If one chooses a suitable integer $\gamma > 0$ and put $m = p^\gamma - 1$, $e_\alpha$ can be explicitly given by a $p$-adic limit:

\[
e_\alpha = \lim_{n \to \infty} T_0(p)^{p^{\gamma m}} \text{ in } \mathcal{H}_{k, u}(Np^\alpha; \mathfrak{O}).
\]
By the commutativity of (3.5), if \( \alpha \geq \beta > 0 \), \( e_\alpha \) is sent to \( e_\beta \) under the projection map (3.6a). Thus (3.6a) induces a surjective \( \mathcal{O} \)-algebra homomorphism (3.6b)

\[
\mathcal{H}_{k, u}(Np^\alpha; \mathcal{O}) \to \mathcal{H}_{k, u}(Np^\beta; \mathcal{O}).
\]

We now take the projective limit of the morphisms (3.6a, b)

\[
\mathcal{H}_{k, u}(Np^\infty; \mathcal{O}) = \lim_{\leftarrow \alpha} \mathcal{H}_{k, u}(Np^\alpha; \mathcal{O}),
\]

\[
\mathcal{H}_{k, u}^\text{ord}(Np^\infty; \mathcal{O}) = \lim_{\leftarrow \alpha} \mathcal{H}_{k, u}^\text{ord}(Np^\alpha; \mathcal{O}).
\]

Put \( U_F(Np^\alpha) = U_f(Np^\alpha) \cap F_f^\times = V_f(Np^\alpha) \cap F_f^\times \) and

\[
\text{Cl}_F(Np^\alpha) = F_\Lambda^\times / F_f^\times U_f(Np^\alpha) F_f^\times \,, \quad \text{Cl}_F(Np^\alpha) = \text{Cl}_F(Np^\alpha) F_f^\times.
\]

Via the correspondence: \( x \mapsto x^{-1} \) for \( x \in F_f^\times \) with \( x_{Np} = 1 \), we can identify these groups with the narrow or usual ray class group of \( F \) modulo \( Np^\alpha \). We shall define \( p \)-profinite groups \( Z = Z(N) = \lim_{\leftarrow \alpha} \text{Cl}_F(Np^\alpha) \), \( \tilde{Z} = \tilde{Z}(N) = \lim_{\leftarrow \alpha} \text{Cl}_F(Np^\alpha) \), and we denote by \( Z_\alpha \) (resp. \( \tilde{Z}_\alpha \)) the kernel of the projection map \( Z \to \text{Cl}_F(Np^\alpha) \) (resp. \( \tilde{Z} \to \text{Cl}_F(Np^\alpha) \)).

We shall define a continuous character: \( \tilde{Z} \to \mathcal{H}_{k, u}(Np^\alpha; \mathcal{O}) \) for \( \alpha = 1, 2, \ldots, \infty \) in order to regard Hecke algebras as algebras over the continuous group algebra of \( \tilde{Z} \). To do this, let \( \Pi(M) \) be the set of all integral ideals prime to \( M \) for each ideal \( M \) of \( \mathfrak{t} \). Then we have a natural map: \( \Pi(Np) \to Z \) with dense image. (There might be a non-trivial kernel of the map: \( \Pi(Np) \to Z \); so we shall correct the statement in [13], page 140, line 9 from the bottom as follows: “the set \( \mathcal{S}(Np) \) is a dense subset” should read “the image of the set \( \mathcal{S}(Np) \) is a dense subset”, where we have written \( \mathcal{S}(Np) \) instead of \( \Pi(Np) \) here. This error in [13] does not affect the result obtained there.) We denote by \( [a] \) the image of \( a \in \Pi(Np) \) in \( Z \). If we write \( U_F(Np^\infty) = \{ x \in U_f(N) \mid x_p = 1 \} \), then we have a natural isomorphism

\[
Z(N) = F_\Lambda^\times / F_f^\times U_f(Np^\infty) F_f^\times 
\]

where "−" indicates the closure in \( F_f^\times \). If we write \( a = x_t \) for each \( a \in \Pi(Np) \) with \( x \in F_f^\times \) and \( x_{Np} = 1 \), then under the isomorphism (3.8), \( [a] \) corresponds to the class of \( x^{-1} \) on the right-hand side. More generally, if one identifies \( F_\Lambda^\times / F_f^\times F_\Lambda^\times \) with the Galois group over \( F \) of the maximal abelian extension \( F_{ab} \) of \( F \) by class field theory, \( [a] \) for \( a \in \Pi(Np) \) coincides with the Artin symbol of \( a \) on the subfield of \( F_{ab} \) fixed by \( F_f^\times U_f(Np^\infty) F_f^\times / F_f^\times F_\Lambda^\times \). Let \( \mu_{p^\alpha} \) be the group of all \( p^\alpha \)-th roots of unity and put \( \mu_{p^\alpha} = \lim_{\alpha} \mu_{p^\alpha} \). The action of \( Z(N) \) as the Galois group over \( F \) on \( \mu_{p^\alpha} \) gives a continuous character

\[
\chi: Z(N) \to \text{Aut}(\mu_{p^\alpha}) \equiv Z_p^\times
\]

such that \( \chi([a]) = \mathcal{N}_{F/\mathbb{Q}}(a) \). For each \( \xi \in \mathbb{Z} \cdot t \), we write \( \xi = [\xi]t \) for \( [\xi] \in \mathbb{Z} \).
and put $\chi_{\ell} = \chi^{[\ell]}: Z(N) \to Z_{p^\infty}$. The groups $Z$ and $\bar{Z}$ can be decomposed as $Z = W \times Z_{tor}$ and $\bar{Z} = \bar{W} \times \bar{Z}_{tor}$ for $Z_{p^\infty}$-free groups $W$ and $\bar{W}$ and finite groups $Z_{tor}$ and $\bar{Z}_{tor}$. Put $W_a = W \cap Z_a$ and $\bar{W}_a = \bar{W} \cap \bar{Z}_a$. We may then identify $W$ and $\bar{W}$ under the natural map: $Z \to \bar{Z}$. Then for sufficiently large $a$, we have $Z_a = W_a = \bar{W}_a = Z_a$. This assertion holds if $a \geq 1$ when $p$ is an odd prime. We denote by $\Lambda$, $\mathcal{A}$ and $\mathcal{A}'$ the continuous group algebras over $\mathcal{O}$ of $W$, $Z$ and $\bar{Z}$, respectively. Namely, 

$$\Lambda = \lim_{\alpha} \mathcal{O}[W/W_a],$$

$$\mathcal{A} = \lim_{\alpha} \mathcal{O}[\mathcal{C}_F(Np^{\alpha})] \quad \text{and} \quad \mathcal{A}' = \lim_{\alpha} \mathcal{O}[\mathcal{C}_F(Np^{\alpha})].$$

In order to give an $\mathcal{A}'$-algebra structure on the Hecke algebras $k_{k, w}(Np^{\alpha}; \mathcal{O})$ ($\alpha = 1, 2, \ldots, \infty$), we let $F_{\alpha}^\infty$ act on $S_{k, w, f}(Np^{\alpha}; B; C)$ by $f \mapsto f|_{k, w}a$ for $a \in F_{\alpha}^\infty$. This action coincides with the operator $[U(N) a U(N)]$ and thus gives an action of $F_{\alpha}^\infty/U_F(Np^{\alpha})F_{\alpha}^\infty$. Since $f(ax) = f(x)$ and $f|_{k, w}a_{\infty} = f$ by (2.4a) for $a \in F_{\alpha}^\infty$, we have that

$$(*) \quad f|_{k, w}a f = f|_{k, w}a = a^{2w - k}f = a^{n + 2v}f.$$

For $a \in I(Np)$, by taking $a \in F_{\alpha}^\infty$ with $a_\tau = a$ and $a_{NP} = 1$, we define $\langle a \rangle_n \in \text{End}_C(S_{k, w, f}(Np^{\alpha}; B; C))$ by

$$f|_{k, w}a_{\infty} = \chi_{-n - 2v}(a)f|_{k, w}a.$$ 

Then ($*$) shows that if $a$ is trivial in $\mathcal{C}_F(Np^{\alpha})$, then $f|_{k, w}a_{\infty} = f$; namely, the finite group $\mathcal{C}_F(Np^{\alpha})$ acts on $S_{k, w, f}(Np^{\alpha}; C)$ via $f \mapsto f|_{k, w}a_{\infty}$. As an operator on $S_{k, w, f}(Np^{\alpha}; C)$, $T(a, a) = \chi_{n + 2v}(a)\langle a \rangle_n$. 

Now we shall show that $T(a, a)$ and $T_0(a, a)$ is contained in $k_{k, w}(Np^{\alpha}; A)$ for arbitrary $A$ (if $n \geq 0$). Since $\langle a \rangle_n$ depends only on the class of $a$ in $\mathcal{C}_F(Np^{\alpha})$, we can choose two prime ideals $\ell$ and $q$ of $F$ such that $\mathcal{N}_{F/Q}(\ell)$ is prime to $\mathcal{N}_{F/Q}(q)$ but $\langle \ell \rangle_n = \langle q \rangle_n$. Then we can find integers $x, y$ such that

$$x\chi_{n + 2v + f}(\ell) + y\chi_{n + 2v + f}(q) = 1.$$ 

Then by (2.9d), we have that

$$\langle \ell \rangle_n = x \chi(\ell)T(\ell, \ell) + y \chi(q)T(q, q)$$

$$= x[T(\ell^2) - T(\ell^2)] + y[T(q^2) - T(q^2)] \in k_{k, w}(Np^{\alpha}; A).$$

Thus for any $a \in I(Np)$, $\langle a \rangle_n \in k_{k, w}(Np^{\alpha}; A)$ and thus

$$T(a, a) = \chi_{n + 2v}(a)\langle a \rangle_n \in k_{k, w}(Np^{\alpha}; A),$$

$$T_0(a, a) = \{ a^{-2v} \} \chi_{n + 2v}(a)\langle a \rangle_n \in k_{k, w}(Np^{\alpha}; A)$$

because $\{ a^{-2v} \} \chi_{n + 2v}(a)$ is always an element in $A$. 

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We now know from (*) and (3.9) that the correspondence: \( \operatorname{II}(Np) \ni a \mapsto T(a, a) \in k_{k, u}(Np^\alpha; \emptyset) \) factors through the image of \( \operatorname{II}(Np) \) in \( \overline{Z(N)} \) and is continuous under the topology induced by \( \overline{Z(N)} \). Then, by continuity, this homomorphism of the semi-group \( \operatorname{II}(N) \) extends to a continuous character: \( \overline{Z(N)} \to k_{k, u}(Np^\alpha; \emptyset) \). The image of \( z \in Z(N) \) or \( a \in \operatorname{II}(Np) \) under this character will be written as \( \langle z \rangle \) or \( \langle a \rangle \in k_{k, u}(Np^\alpha; \emptyset) \) (\( \alpha = 1, 2, \ldots \)). This character is naturally compatible with the projection map: \( k_{k, u}(Np^\alpha; \emptyset) \to k_{k, u}(Np^\beta; \emptyset) \) for \( \alpha \geq \beta > 0 \). By taking the projective limit, we have a continuous character: \( \overline{Z(N)} \to k_{k, u}(Np^\infty; \emptyset) \). By the universality of the continuous group algebra, \( k_{k, u}(Np^\alpha; \emptyset) \) becomes an algebra over \( \mathcal{A} \) and \( \Lambda \) via this character for \( \alpha = 1, 2, \ldots, \infty \).

**Theorem 3.2.** For each \( k \in \mathbb{Z} \cdot t \ (t = \sum \sigma \alpha) \) with \( k \geq 2t \) (i.e. \( v = 0 \) and \( n \geq 0 \)), there exists an \( \mathcal{A} \)-algebra isomorphism:

\[
k_{2t, 1}(Np^\infty; \emptyset) \cong k_{k - 1}(Np^\infty; \emptyset),
\]

which takes \( T(n) \) to \( T(n) \) for all ideals \( n \) of \( \mathfrak{r} \).

The implication of this theorem in terms of \( p \)-adic modular forms will be explained in Section 5. The same type of assertion is also expected to be true for \( k_{k, u}(Np^\infty; \emptyset) \) even for general \( k \in \mathbb{Z}[I] \) (or \( v \in \mathbb{Z}[I] \)). The author hopes to come back to this problem in the future. As for the ordinary part \( k_{k, u}^{\text{ord}}(Np^\infty; \emptyset) \), we have the following general result:

**Theorem 3.3.** If \( k \) and \( k' \) in \( \mathbb{Z}[I] \) satisfy \( k \geq k' \geq 2t \) and \( k \sim k' \sim -2v \), then there exists an \( \mathcal{A} \)-algebra isomorphism:

\[
k_{k, u}(Np^\infty; \emptyset) \cong k_{k', u}(Np^\infty; \emptyset)
\]

which takes \( T_0(n) \) to \( T_0(n) \) for all ideals \( n \), where \( w = v + k - t \) and \( w' = v + k' - t \). Moreover, \( k_{k, u}^{\text{ord}}(Np^\infty; \emptyset) \) is a torsion-free \( \Lambda \)-module of finite type.

Theorems 3.2 and 3.3 will be proved in Section 11 after the analysis of the structure of cohomology groups in Sections 8, 9 and 10, where we shall employ Shimura’s method in [30] as a key technique. Since \( k_{k, u}^{\text{ord}}(Np^\infty; \emptyset) \) only depends on \( v \mod \mathbb{Z} \cdot t \ (w = v + k - t) \), we hereafter write \( k_{v}^{\text{ord}}(N; \emptyset) \) for \( k_{k, u}^{\text{ord}}(Np^\infty; \emptyset) \).

Let \( \lambda: W \to Q_p^x \) be a continuous character. Then we can extend \( \lambda \) to an algebra homomorphism \( \lambda: \Lambda \to Q_p^x \). Let \( P_\lambda \) denote the point of \( \text{Spec} (\Lambda)/_\emptyset (Q_p^x) \) corresponding to this algebra homomorphism. For each finite order character \( \varepsilon: W \to Q_p^x \) and \( m \in \mathbb{Z} \cdot t \), we write \( P_{m, \varepsilon} \) for \( P_{X_m \varepsilon} \). We also define, if \( \varepsilon \) factors through \( W/W_\alpha \) and \( W_\alpha = Z_\alpha \),

\[
S_{k, w, \varepsilon}(Np^\alpha; \emptyset; C) = \left\{ f \in S_{k, w, \varepsilon}(Np^\alpha; \emptyset; C) \mid f \langle w \rangle_n = \varepsilon(w) f \right\}
\]

Here, note that the action: \( W \ni w \mapsto \langle w \rangle_n \) factors through the finite group
$W/W_a$. For any $\mathcal{O}(v)$-algebra $A$ in $\mathbf{C}$ containing the values of $\varepsilon$, we define $\mathfrak{h}_{k,w}(Np^a, \varepsilon; A)$ to be the $A$-subalgebra of $\text{End}_\mathbf{C}(S_{k,w}(Np^a, \varepsilon; \mathbf{C}))$ generated over $A$ by $T_0(\pi)$ for all ideals $\pi$. Let $\mathcal{O}(v, \varepsilon)$ be the valuation ring of the field generated over $\Phi(v)$ by the values of $\varepsilon$ corresponding to the embedding: $\mathbb{Q} \to \overline{\mathbb{Q}}_p$. If $\varepsilon$ has values in $\mathcal{O}$, $\mathcal{O}$ contains $\mathcal{O}(v, \varepsilon)$. We then put

$$\mathfrak{h}_{k,w}(Np^a, \varepsilon; \mathcal{O}) = \mathfrak{h}_{k,w}(Np^a, \varepsilon; \mathcal{O}(v, \varepsilon)) \otimes_{\mathcal{O}(v, \varepsilon)} \mathcal{O}.$$ 

There is a natural surjection: $\mathfrak{h}_{k,w}(Np^a; \mathcal{O}) \to \mathfrak{h}_{k,w}(Np^a, \varepsilon; \mathcal{O})$ which sends $T_0(\pi)$ to $T_0(\pi)$. Let $K$ be the quotient field of $\mathcal{O}$. Then, we can define

$$\mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; \mathcal{O}) = e_{\alpha} \mathfrak{h}_{k,w}(Np^a, \varepsilon; \mathcal{O}),$$

$$\mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; K) = \mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; \mathcal{O}) \otimes_{\mathcal{O}} K.$$

By the following theorem, $\mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; K)$ and hence $\mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; \mathcal{O})$ is independent of the choice of $\alpha$ such that $\varepsilon$ factors through $W/W_a$ and $W_a = Z_a$:

**Theorem 3.4.** Let $n$ be an element of $\mathbb{Z}[I]$ with $n \sim -2v$ and $n \geq 0$, and let $\varepsilon: W/W_a \to \mathcal{O}_\mathbf{X}$ be a finite order character. Suppose that $W_a = Z_a$. Write $P$ for $P_n + 2v, \varepsilon$, and let $\Lambda_P$ denote the localization of $\Lambda$ at $P$. Then there is a canonical isomorphism:

$$\mathfrak{h}^{\text{ord}}_\varepsilon(N; \mathcal{O}) \otimes_{\Lambda} \Lambda_P/\Lambda_P = \mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; K)$$

for $k = n + 2t$ and $w = v + k - t$,

which takes $T_0(\pi)$ to $T_0(\pi)$ for all $\pi$. Especially, $\mathfrak{h}^{\text{ord}}_\varepsilon(N; \mathcal{O}) \otimes_{\Lambda} \Lambda_P$ is free of finite rank over $\Lambda_P$, and the dimension of $\mathfrak{h}_{k,w}^{\text{ord}}(Np^a, \varepsilon; K)$ over $K$ is independent of $\varepsilon$ and $n$ (or $k$) and is equal to the dimension of $\mathfrak{h}^{\text{ord}}_\varepsilon(N; \mathcal{O}) \otimes_{\Lambda} \mathcal{L}$ over $\mathcal{L}$ for the quotient field $\mathcal{L}$ of $\Lambda$.

This theorem will be proved in Section 12. Let us now indicate an important implication of the theorem and the duality theorem (Th. 5.3 below). Let $\mathcal{L}$ be the quotient field of $\Lambda$. We fix an algebraic closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ and consider $\overline{\mathbb{Q}}_p$ as a subfield of $\overline{\mathcal{L}}$. We shall fix a $\Lambda$-algebra homomorphism $\lambda: \mathfrak{h}^{\text{ord}}_\varepsilon(N; \mathcal{O}) \to \overline{\mathcal{L}}$. Let $\mathcal{K}$ be the quotient field of the image of $\lambda$. Then, by Theorem 3.3, $\mathcal{K}$ is a finite extension of $\mathcal{L}$. Let $\mathcal{I}$ be the integral closure of $\Lambda$ in $\mathcal{K}$. Then again by Theorem 3.3, $\lambda$ in fact has values in $\mathcal{I}$. Put $\mathcal{O}_\mathcal{I} = \overline{\mathbb{Q}}_p \cap \mathcal{I}$. Then $\mathcal{O}_\mathcal{I}$ is a valuation ring, finite flat over $\mathcal{O}$. Let $\mathcal{X} = \mathcal{X}(\mathcal{I})$ be the space of all $\overline{\mathbb{Q}}_p$-valued points of $\text{Spec}(\mathcal{I})/\mathcal{O}$. Thus

$$\mathcal{X} = \text{Hom}_{\text{alg}}(\mathcal{I}, \overline{\mathbb{Q}}_p).$$

Let $\mathcal{X}_{\text{alg}}(\Lambda) = \{ P_n, \varepsilon \in \mathcal{X}(\Lambda) | n \in \mathbb{Z} \cdot t, n \geq 0 \text{ and } \varepsilon: W \to \mathbb{Q}_p^\times \text{ be a finite order character} \}$. We have a natural morphism $\pi: \text{Spec}(\mathcal{I})/\mathcal{O} \to \text{Spec}(\Lambda)/\mathcal{O}$ of
schemes over \( \mathcal{O} \). Put \( \mathcal{X}_{\text{alg}}(\mathcal{F}) = \pi^{-1}(\mathcal{X}_{\text{alg}}(\Lambda)) \). If one considers \( P \in \mathcal{X}_{\text{alg}}(\mathcal{F}) \) as an \( \mathcal{O} \)-algebra homomorphism \( P: \mathcal{F} \to \overline{\mathbb{Q}}_p \), then \( P|_{\Lambda} = P_{n, \epsilon} \) for some \( n \in \mathbb{Z} \cdot t \) and a finite order character \( \epsilon: \mathbb{W} \to \overline{\mathbb{Q}}^{\times} \). We write this \( n \) as \( n(P) \) and this \( \epsilon \) as \( \epsilon_p \). The minimum of \( \alpha \) such that \( \text{Ker}(\epsilon_p) \supset \mathbb{Z}_\alpha \) will be denoted by \( \alpha(P) \). For each \( P \in \mathcal{X}_{\text{alg}}(\mathcal{F}) \), regarding \( P \) as an \( \mathcal{O} \)-algebra homomorphism \( P: \mathcal{F} \to \overline{\mathbb{Q}}_p \), we can consider an \( \mathcal{O} \)-algebra homomorphism \( \lambda_p: \mathcal{H}_{v}(N; \mathcal{O}) \to \overline{\mathbb{Q}}_p \) given by \( \lambda_p = P \circ \lambda \). Then \( \lambda_p \) factors through

\[
\mathcal{H}_{k, w}^{\text{ord}}(Np^{\alpha(P)}, \epsilon_p; K) \cong \mathcal{H}_{V}(N; \mathcal{O}) \otimes_{\Lambda} \Lambda_{\overline{P}}/\overline{P} \Lambda_{\overline{P}}
\]

for \( \overline{P} = P|_{\Lambda} \) by Theorem 3.4 and can be considered as a \( K \)-algebra homomorphism

\[
\lambda_p: \mathcal{H}_{k, w}^{\text{ord}}(Np^{\alpha(P)}, \epsilon_p; K) \to K
\]

for \( k = n(P) - 2v + 2t \) and \( w = -v + n(P) + t \).

Then we have:

**Corollary 3.5.** For each \( P \in \mathcal{X}_{\text{alg}}(\mathcal{F}) \) with \( n(P) \geq 2v \), the value \( \lambda_p(T(n)) \) is contained in \( \overline{\mathbb{Q}} \) for all \( n \), and since \( \lambda_p(T(n)) \) is a complex number by the fixed embedding of \( \overline{\mathbb{Q}} \) into \( C \), there is a non-zero cusp form \( f_\mathcal{P} \in S_{k, w, l_p}(Np^{\alpha(P)}, \epsilon_p; C) \) for \( k = n(P) - 2v + 2t \) and \( w = -v + n(P) + t \) such that \( f_\mathcal{P}|T(n) = \lambda_p(T(n))f_\mathcal{P} \) for all ideals \( n \). The cusp form \( f_\mathcal{P} \) is uniquely determined by the above conditions up to constant factors. Conversely, suppose that there is a common eigenform \( f \) of all Hecke operators \( T(n) \) in \( S_{k, w, l_p}(Np^{\alpha}; C) \) whose eigenvalue for \( T_\mathcal{P}(p) \) is a \( p \)-adic unit in \( \overline{\mathbb{Q}}_p \) and whose weight \( k \geq 2t \) satisfies \( k \sim 2v \). Then there exists a \( \Lambda \)-algebra homomorphism \( \lambda: \mathcal{H}_{v}(N; \mathcal{O}) \to \mathcal{F} \) and \( P \in \mathcal{X}_{\text{alg}}(\mathcal{F}) \) such that

\[
f|T(n) = \lambda_p(T(n))f \quad \text{for all } n \text{ prime to } p.
\]

We shall prove in Section 5 a slightly stronger result than the statement of Corollary 3.5 after proving duality theorems between Hecke algebras and the space of cusp forms.

Let \( \lambda: \mathcal{H}_{v}(N; \mathcal{O}) \to \mathcal{F} \) be a \( \Lambda \)-algebra homomorphism. We consider the set:

\[
\{ \lambda': \mathcal{H}_{v}(N'; \mathcal{O}) \to \mathcal{F} \mid \lambda'(T_0(\ell)) = \lambda(T_0(\ell)) \}
\]

except for finitely many prime ideals \( \ell \),

where the level \( N' \) varies. Obviously in this set, there exists a \( \Lambda \)-algebra homomorphism \( \lambda_0: \mathcal{H}_{v}(C; \mathcal{O}) \to \mathcal{F} \) with the smallest level \( C \). This \( \lambda_0 \) is called primitive or a primitive homomorphism associated with \( \lambda \). The level \( C \) is called the conductor of \( \lambda \). We can make an analogous definition in the finite level case: Let \( \phi: \mathcal{H}_{k, w}(M; \mathcal{O}) \to \overline{\mathbb{Q}}_p \) be an \( \mathcal{O} \)-algebra homomorphism for an \( \mathcal{O} \)-ideal \( M \). We
consider the set
\[ \{ \phi': \mathcal{H}_{k, w}(M'; \emptyset) \to \overline{Q}_p | \phi(T(\ell)) = \phi'(T(\ell)) \text{ for almost all prime ideals } \ell \}. \]

Then in this set, we can find a unique \( \phi_0: \mathcal{H}_{k, w}(C; \emptyset) \to \overline{Q}_p \) with the smallest level \( C \). This \( \phi_0 \) is called primitive or the primitive homomorphism associated with \( \phi \). The level \( C \) is called the conductor of \( \phi \), which is a divisor of \( M \). This fact can be deduced from a result of Miyake [23] in view of the equivalence between the following two conditions:

(3.10a) \( \phi \) is primitive of conductor \( C \),

(3.10b) the cusp form \( f \) in \( S_{k, w}(C; C) \) satisfying \( f|T(n) = \phi_0(T(n))f \) for all \( n \) is a new form of exact level \( C \) in the sense of [23].

Note that the existence of \( f \) as in (3.10b) will be guaranteed by Theorem 5.3 in Section 5.

We have decomposed \( \overline{Z}(N) = W \times \overline{Z}_{\text{tor}} \). Let \( \psi: \overline{Z}_{\text{tor}} \to \overline{Q} \) be the combination: \( \overline{Z}_{\text{tor}} \hookrightarrow \overline{Z}(N) \to \mathcal{H}^{\text{ord}}_v(N; \emptyset) \to \overline{\mathcal{L}} \). Then \( \psi \) can be considered to be a finite order character of \( F_{\lambda}^x/F_{\infty}^xF_{\infty}^+ \). This character \( \psi \) will be called the character of \( \lambda \). Similarly, for each homomorphism \( \phi: \mathcal{H}_{k, w}(M, \emptyset) \to \overline{Q}_p \), the correspondence: \( \Pi(M) \ni \alpha \to \phi(\langle \alpha \rangle_n) \in \overline{Q} \) gives a character of \( \text{Cl}_f(M) \). This character is called the character of \( \phi \). By Corollary 3.5 and (3.9), we have:

(3.11) The character of \( \lambda_p \) (\( P \in \mathcal{X}_{\text{alg}}(\mathcal{I}) \)) is given by \( \epsilon_p \psi \cdot (\omega \circ \chi_{-n(P)}) \),

where \( \omega: Z_p^x \to \overline{Q}^x \) is the Teichmüller character. Then we have:

**Theorem 3.6.** Let \( \lambda: \mathcal{H}^{\text{ord}}_v(N; \emptyset) \to \overline{\mathcal{L}} \) be a \( \Lambda \)-algebra homomorphism. Then the primitive homomorphism \( \lambda_0 \) associated with \( \lambda \) is unique, and its conductor \( C \) is a divisor of \( N \). If \( \lambda \) itself is primitive, then for all \( P \in \mathcal{X}_{\text{alg}}(\mathcal{I}) \), the conductor of \( \lambda_p = P \circ \lambda \) is divisible by \( N \). If the conductor of \( \lambda_p \) is moreover divisible by every prime factor of \( p \), then \( \lambda_p \) itself is primitive.

**Corollary 3.7.** Let \( \lambda: \mathcal{H}^{\text{ord}}_v(N; \emptyset) \to \overline{\mathcal{L}} \) be a primitive homomorphism, and let \( \mathcal{K} \) be the quotient field of the image of \( \lambda \). Then, we can decompose

\[ \mathcal{H}^{\text{ord}}_v(N; \emptyset) \otimes_\Lambda \mathcal{K} = \mathcal{K} \oplus \mathcal{B} \]

as an algebra direct sum so that the projection to the first factor \( \mathcal{K} \) coincides with \( \lambda \) on \( \mathcal{H}^{\text{ord}}_v(N; \emptyset) \).

Theorem 3.6 together with Corollary 3.7 will be proved in Section 12. Now we touch briefly on the module of congruence and the module of differentials attached to \( \lambda \), which have an intimate relation with a certain \( L \)-function of cusp forms at least in the case: \( F = Q \) as disclosed in [12], [14] and [15]. Let us define \( \lambda: \mathcal{H}^{\text{ord}}_v(N; \emptyset) \otimes_\Lambda \mathcal{I} \to \mathcal{I} \) by the combination of \( \lambda \otimes \text{id}: \mathcal{H}^{\text{ord}}_v(N; \emptyset) \otimes_\Lambda \mathcal{I} \to \mathcal{I} \otimes_\Lambda \mathcal{I} \) with the multiplication map: \( \mathcal{I} \otimes_\Lambda \mathcal{I} \to \mathcal{I} \). Suppose \( \lambda \) to be primitive, and let \( \mathcal{H}^{\text{ord}}_v(N; \emptyset) \otimes_\Lambda \mathcal{K} = \mathcal{K} \oplus \mathcal{B} \) be the decomposition as in Corollary 3.7. Let
\[ \text{pr}_{\mathcal{A}}: h^{\text{ord}}(N; \mathcal{O}) \otimes \lambda \mathcal{I} \to \mathcal{B} \] is the projection map, and define
\[ \delta: h^{\text{ord}}(N; \mathcal{O}) \otimes \lambda \mathcal{I} \to \mathcal{I} \oplus \text{pr}_{\mathcal{A}}(h^{\text{ord}}(N; \mathcal{O}) \otimes \lambda \mathcal{I}) \]
by the diagonal map of \( \lambda \) and \( \text{pr}_{\mathcal{A}} \). The module of congruence \( \mathcal{C}_0(\lambda) \) is defined by
\[ \mathcal{C}_0(\lambda) = \text{Coker}(\delta). \]
The module of differentials \( \mathcal{C}_1(\lambda) \) is defined by
\[ \mathcal{C}_1(\lambda) = \Omega^1_{h/\mathcal{I}} \otimes_h \mathcal{I}, \]
where we have written simply \( h \) for \( h^{\text{ord}}(N; \mathcal{O}) \otimes \lambda \mathcal{I} \) and \( \mathcal{I} \) is regarded as an \( h \)-module via \( \lambda: h \to \mathcal{I} \).

**Corollary 3.8.** The modules \( \mathcal{C}_0(\lambda) \) and \( \mathcal{C}_1(\lambda) \) are torsion \( \mathcal{I} \)-modules of finite type. Moreover, we have the identity of support of these modules:
\[ \text{Supp}_{\mathcal{I}}(\mathcal{C}_0(\lambda)) = \text{Supp}_{\mathcal{I}}(\mathcal{C}_1(\lambda)). \]

The author hopes to clarify the relation of these modules with certain \( p \)-adic \( L \)-functions of cusp forms on a future occasion.

**Proof.** Since \( h = h^{\text{ord}}(N; \mathcal{O}) \otimes \lambda \mathcal{I} \) is of finite type over \( \mathcal{I} \), the first assertion is obvious. Write \( \mathcal{I} \) for \( \text{pr}_{\mathcal{A}}(h) \), and let \( a \) be the kernel of \( \lambda \). Then, from [13, Lemma 1.1], we know that \( \mathcal{C}_1(\lambda) \cong a/a^2 \). We take the intersection \( a_0 = \text{Im}(\delta) \cap \mathcal{I} \) inside \( \mathcal{I} \oplus \mathcal{I} \). Then \( \delta \) induces a surjection: \( a \to a_0 \), and hence we have a surjection: \( \mathcal{C}_1(\lambda) \to a_0/a_0^2 \). Note that
\[ C_0(\lambda) = (\mathcal{I} \oplus \mathcal{I})/\text{Im}(\delta) = \mathcal{I} + \text{Im}(\delta)/\text{Im}(\delta) \cong \mathcal{I}/\mathcal{I} \cap \text{Im}(\delta) = \mathcal{I}/a_0. \]
This shows that \( \text{Supp}(\mathcal{C}_1(\lambda)) \supset \text{Supp}(\mathcal{C}_0(\lambda)) \). If we have a vanishing
\[ \mathcal{C}_0(\lambda)_P = \mathcal{C}_0(\lambda) \otimes_{\mathcal{I}} \mathcal{I}_P = 0 \]
for the localization \( \mathcal{I}_P \) of \( \mathcal{I} \) at a prime ideal \( P \), then we have a decomposition:
\[ h_P = h \otimes_{\mathcal{I}} \mathcal{I}_P = \mathcal{I}_P \oplus (\mathcal{I} \otimes_{\mathcal{I}} \mathcal{I}_P) \]
Thus \( \mathcal{I}_P \) is \( h_P \)-projective and \( \mathcal{C}_1(\lambda) \otimes_{\mathcal{I}} \mathcal{I}_P = \text{Tors}^1_{\mathcal{I}_P}(\mathcal{I}_P, \mathcal{I}_P) = 0 \) by [13, Lemma 1.1]. This shows that \( \text{Supp}(\mathcal{C}_0(\lambda)) \supset \text{Supp}(\mathcal{C}_1(\lambda)) \), which finishes the proof.

Let us give here a few words about the filtrations \( \overline{Z}_\alpha \subset \overline{Z} \) and \( Z_\alpha \subset Z \). There is a commutative diagram of natural maps:
\[
\begin{array}{cccccc}
0 & \longrightarrow & Z_\alpha & \longrightarrow & Z & \longrightarrow & \text{Cl}_F(Np^\alpha) & \longrightarrow & 0 \\
\downarrow a & & \downarrow b & & \downarrow c & & & & \\
0 & \longrightarrow & \overline{Z}_\alpha & \longrightarrow & \overline{Z} & \longrightarrow & \overline{\text{Cl}}_F(Np^\alpha) & \longrightarrow & 0.
\end{array}
\]
Since the morphisms $b$ and $c$ are surjective and their kernels are killed by a power of $2$, $a$ is an isomorphism if $\alpha \geq 1$ and $p > 2$. Since the kernels of $b$ and $c$ are generated by the image of $F^s_\infty$, the morphism $a$ is always surjective. The finiteness of the kernel of $b$ implies that $a$ gives a surjective isomorphism for sufficiently large $\alpha$ even if $p = 2$. Especially, $a$ induces a surjective isomorphism between the $Z_p$-torsion-free parts of $Z$ and $\bar{Z}$.

4. Stability of integral cusp forms under Hecke operators

Throughout this section and the next, unless otherwise mentioned, we suppose that $B = M_{2}(F)$. We shall give an exposition of two methods for proving stability under Hecke operators $T_T(A)$ of the space of rational or even integral cusp forms. One is due to Shimura’s theory of canonical models [31] and [34], and the other is derived from the moduli theoretic interpretation of integral modular forms and the $q$-expansion principle studied by Deligne and Ribet [3], Rapoport [27] and Katz [19]. We shall do this because the former method may be generalized to a vast class of algebraic groups for which canonical models exist and the latter gives a stronger result concerning integral cusp forms.

Firstly, we shall define rational subspaces of $S^*_{k, \text{w. } I}(N; C)$ for each ideal $N$ of $\mathfrak{m}$. Let $A$ be a subalgebra of $C$ satisfying (3.1) for the fixed $v \in Z[I]$. For each ideal $a$, we choose a generator $\{a^v\}$ of $a^vA$ as in Section 3. Then the symbol $\{a^v\}$ satisfies the multiplicative relation: $\{a^\epsilon\}\{d^\epsilon\} = \{(ad)^\epsilon\}$. For $\xi \in F^x$, we write $\{\xi^v\}$ for $\{(\xi^t)^v\}$. We take $n, k, w \in Z[I]$ satisfying $0 \leq n - 2v$, $k = n + 2t \geq 2t$, $w = v + k - t$ and $\hat{w} = k - w = t - v$. For each weight $\eta \in Q \cdot t$, we write $\eta = [\eta]t$ for $[\eta] \in Q$. For the symbols which are not defined here, see the introduction.

**Proposition 4.1.** Let $a'$ be the different of $F/Q$, and put

$$F^x = \{\xi \in F^x | \xi^v > 0 \text{ for all } \sigma \in I\}.$$

Then each element $f \in S^*_{k, \text{w. } I}(N; C)$ has the Fourier expansion of the following form:

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y_f|_A y_\infty^w \sum_{\xi \in F^x} a(\xi y a', f) \{((\xi y a')^v) \xi^{-v} e_F(\sqrt{-1} \xi y_\infty) e_F(\xi x),$$

where the function: $a \mapsto a(a, f) \in C$ is a function on the group of fractional ideals of $F$ which vanishes unless $a$ is integral.

**Proof.** For each $f$ as in the proposition, we define another function $f_0: G_\Lambda \to C$ by

$$f_0(g) = |\det(g)|_A^{w - (k/2)} f(g).$$
Then we can easily check that \( f_0 \) is contained in \( S_{k^*}; k/2, t(N; C) \), and by [34, (2.18)], \( f_0 \) has the (unique) Fourier expansion of the form:

\[
 f_0\left( \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \right) = \sum_{\xi \in \mathcal{F}_{k^*}} a_0(\xi y_d, f) \xi^{k/2} y_{\infty}^{k/2} e_F(i \xi y_{\infty}) e_F(\xi x),
\]

where \( \langle a \mapsto a_0(a, f) \rangle \in \mathbb{C} \) is a function on fractional ideals of \( F \) vanishing outside integral ideals. If we put

\[
 (4.1) \quad a(a, f) = a_0(a, f) \mathcal{N}_{|\mathcal{F}|/Q}(ad^{-1})^{-1} - (|k/2| - w) \{ a^{-v} \}
\]

where \( \langle (4.2) \rangle \), we obtain the result.

The following corollary can be deduced from [34, (2.23)] by (4.1) or from an easy calculation by use of an explicit decomposition of \( \mathcal{F}(n) \) into the disjoint union of left cosets of \( U_1(N) \):

**Corollary 4.2.** For each \( f \in S_{k^* ; k/2, t(N; C)} \),

\[
 (4.2) \quad a(m, f | T_0(n)) = \sum_{\ell | m, \ell | n \ell + N = t} \mathcal{N}_{|\mathcal{F}|/Q}(\ell) a(mn/\ell^2, f | T_0(\ell', \ell'))
\]

\[
 = \sum_{\ell | m, \ell | n \ell + N = t} \mathcal{N}_{|\mathcal{F}|/Q}(\ell) \{ \ell^{2v} \}^{-1} a(mn/\ell, f | t(\ell')).
\]

For a while, we shall deal with a general quaternion algebra \( B \) unramified at every finite place not necessarily equal to \( M_2(F) \). We fix a complete representative set \( \{ a_i \}_{i=1, \ldots, h} \) for \( \mathcal{F}_i(1) = F^\times_i / F^\times_i \otimes F^\times_\infty \) such that \( a_i \in F^\times_i \cap i \) and \( (a_i)_N = 1 \). We define \( t_i \in \text{GL}_2(F_f) \) by \( t_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \). Then, if \( B \) is indefinite, we have a disjoint decomposition:

\[
 G_B^B = \prod_{i=1}^h G_B^B t_i^{-1} U_i(N) G_B^\infty = \prod_{i=1}^h G_B^B t_i V_i(N) G_B^\infty.
\]

When \( B \) is totally definite, we just choose \( \{ t_i \}_{i=1, \ldots, h} \) in \( G_f^B \) which satisfies the above type of decomposition; so, in this case, \( h \) may be different from the narrow ray class number of \( F \). Then, we know if \( B \) is indefinite,

\[
 G_A^B = \prod_{i=1}^h G_Q^B t_i x V_i(N) x^{-1} G^B_\infty \quad \text{for all } x \in G_A^B.
\]

Put

\[
 (4.3) \quad \Gamma_i(N) = t_i^{-1} U_E(N) G_\infty^B, t_i \cap G_Q^B = t_i V_E(N) G_\infty^B, t_i^{-1} \cap G_Q^B.
\]
Now we return to the analysis of $B = M_g(F)$. Then by (2.6a), we have a canonical isomorphism:

$$S_{k, w, I}(N; C) \cong \bigoplus_{i=1}^{h} S_{k, \tilde{w}, I}(\Gamma_{k}^{\tilde{w}}(N); C).$$

For each $f \in S_{k, w, I}(N; C)$, we denote $f_i$ by $f_i$ as in (2.4b), which is an element in $S_{k, \tilde{w}, I}(\Gamma_{k}^{\tilde{w}}(N); C)$. Then, by definition of the above isomorphism, we have

**Corollary 4.3.** Let $a_i = a_{i, w}$, and put $a_i^* = a_i^{-1}d^{-1}$ and

$$a_i^* = \{ \xi \in a_i^* | \xi^\sigma > 0 \text{ for all } \sigma \in I \}.$$

Then, for each $f \in S_{k, w, I}(N; C)$, $f_i(z) \in S_{k, \tilde{w}, I}(\Gamma_{k}^{\tilde{w}}(N); C)$ has the following Fourier expansion:

$$f_i(z) = c_{v, i} \sum_{\xi \in a_i^*} a(\xi a_i, f) \{ \xi^v \} \xi^{-v} e_F(\xi z),$$

where $c_{v, i} = N_{F/Q}(a_i)^{-1}\{ (a_i, f)^v \}$.

Here note that $\{ \xi^v \} \xi^{-v}$ is a unit in $A$.

Now we shall define the space of $A$-integral cusp forms. To do this, let us prepare some notation. For each ideal $a$, we put

$$a_+ = \{ \xi \in a | \xi^\sigma > 0 \text{ for all } \sigma \in I \}.$$

Then we consider the formal power series ring

$$A[[q]]_a = \left\{ \sum_{\xi \in a_+ \cup \{0\}} a(\xi)q^\xi \middle| a(\xi) \in A \right\}.$$

Especially, we write $A[[q]]_a$ for $A[[q]]_{a^*_+}$. Then, replacing $e_F(\xi z)$ by $q^\xi$, we have an embedding for each congruence subgroup $\Gamma$ of $GL_2^+(F) = GL_2(F) \cap G_{\infty}$:

$$S_{k, w, I}(\Gamma; C) \hookrightarrow C[[q]]_a$$

for a suitable choice of $a$. By this embedding, we shall regard $S_{k, w, I}(\Gamma; C)$ as a subspace of $C[[q]]_a$. We then put

$$S_{k, w, I}(\Gamma; A) = A[[q]]_a \cap S_{k, w, I}(\Gamma; C), \quad S_{k, w, I}(A) = \lim_{\Gamma \to \Gamma} S_{k, w, I}(\Gamma; A).$$

Then, for any $\alpha \in GL_2^+(F)$, $f \mapsto f|_{k, w, \alpha}$ gives an endomorphism of $S_{k, w, I}(C)$, where

$$(4.4) \quad (f|_{k, w, \alpha})(z) = \det(\alpha)^w j_l(\alpha, z)^{-k} f(\alpha(z)).$$
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As for the spaces of functions on $G_A$, we define

$$(4.5a) \quad S_{k, w, l}(N; A) = S_{k, w, l}(N; M_2(F); A)$$

$$= \{ f \in S_{k, w, l}(N; C) \mid a(a, f) \in A \text{ for all } a \in \Pi(1) \}.$$ 

Then the isomorphism (2.6a) induces

$$(4.5b) \quad S_{k, w, l}(N; A) \cong \bigoplus_{i=1}^h c_{v, i} S_{k, \hat{w}, l}(\Gamma_E^i(N); A).$$

Now we shall give an exposition of Shimura's method to prove the stability of $S_{k, s, l}(N; K)$ under Hecke operators for each finite extension $K/\Phi(v)$. Put

$$(4.6a) \quad \mathcal{G}_+ = \{ x \in GL_2(F_A) \mid \det(x) \in F^\times_{\infty} \text{ and } \det(x) \in Q_A^\times F^\times_{\infty+} \}.$$ 

Since $Q_A^\times F^\times_{\infty+}/F^\times_{\infty+} \cong Q_A^\times/Q_A^\times Q_{\infty+} \cong \text{Gal}(Q_{ab}/Q)$ canonically for the maximal abelian extension $Q_{ab}/Q$, we can let $x \in \mathcal{G}_+$ act on $Q_{ab}$ via $\det(x)$. The corresponding element of $\text{Gal}(Q_{ab}/Q)$ to $x \in \mathcal{G}_+$ will be written as $\rho(x)$. Define

$$(4.6b) \quad \mathcal{G}_+(K) = \{ x \in \mathcal{G}_+ \mid \rho(x) \text{ acts trivially on } K \cap Q_{ab} \}.$$ 

One may consider $\rho(x)$ for $x \in \mathcal{G}_+(K)$ to be an element of $\text{Gal}(KQ_{ab}/K)$. It has been shown by [34, Prop. 1.7]:

$$(4.7a) \quad \text{For each extension } K/\Phi(v) \text{ inside } C, \text{ we have a natural isomorphism:}$$

$$S_{k, \hat{w}, l}(K) \cong S_{k, \hat{w}, l}(\Phi(v)) \otimes_{\Phi(v)} K.$$ 

This shows especially that if we define an action of $\sigma \in \text{Gal}(Q/\Phi(v))$ on $f = \sum_{\xi} a(\xi)q^\xi \in \hat{Q}[[q]]_a$ by $f^\sigma = \sum_{\xi} a(\xi)^\sigma q^\xi$, then we have

$$(4.7b) \quad \text{Gal}(\Phi(v)Q_{ab}/\Phi(v)) \text{ acts on } S_{k, \hat{w}, l}(\Phi(v)Q_{ab}) \text{ via } f \mapsto f^\sigma.$$ 

For each positive integer $N$, we put

$$U_N' = G_\infty^+ \times \{ x \in GL_2(\hat{\mathbb{F}}) \mid x \equiv \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \mod NM_2(\hat{\mathbb{F}}) \},$$

$$U_N = \{ x \in U_N' \mid \det(x) \in Q_A^\times \} \cdot G_{\infty+}, \quad \Gamma_N = GL_2(F) \cap U_N.$$ 

Note that for each $\gamma \in \Gamma_N$, $\det(\gamma) \in Q_A^\times$ by definition of $U_N$. This implies that $\Gamma_N$ is a subgroup of $SL_2(F)$, and hence, $\Gamma_N$ is the principal congruence subgroup modulo $N$ of $SL_2(F)$. By the strong approximation theorem (cf. [31, Prop. 3.4]), we know that $\mathcal{G}_+ = U_N' \cdot GL_2^+(F)$. Then, for each $x \in \mathcal{G}_+$, we decompose $x = u\alpha$ for $\alpha \in GL_2^+(F)$ and $u \in U_N$ and consider the correspondence: $x \mapsto \det(\alpha)\hat{w}$. If $x = u\alpha = u'\alpha'$ for another choice of $u' \in U_N$ and $\alpha' \in \text{Gal}(\Phi(v)Q_{ab}/\Phi(v))$.
GL₂(𝔽), then \( u^{-1}u = α α^{-1} \in Γ_N \). Especially, \( \det(α) = \det(α') \). Thus the function: \( x \rightarrow \det(α)^{\check{ω}} \in Φ(τ) \) gives a continuous quasi-character of \( \mathcal{G}_+ \). Now we shall present a slight modification of a result of Shimura:

**Theorem 4.4.** For each finite extension \( K \) of \( Φ(τ) \), there is a continuous right action of \( S_{k, \check{ω}, I}(KQ_{ab}) \) which is characterized by the following properties:

\[(4.8a) \quad (f + g)^x = f^x + g^x, \quad (f^x)^y = f^{xy},\]
\[(4.8b) \quad f^α = f_{|k, \check{ω} α} \quad \text{as in (4.4) if} \quad α \in GL_2^+(𝔽),\]
\[(4.8c) \quad f^x = f^{p(x)} \quad \text{if} \quad x = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad \text{with} \quad t \in \hat{ℤ}^×.\]

Here the continuity of the action means that for each \( f \in S_{k, \check{ω}, I}(KQ_{ab}) \), we can find a positive integer \( N \) so that \( f \in S_{k, 0} I(KQ_{ab}) \) for all \( α \in Gal(KQ_{ab}/K) \).

It might be necessary to explain how to deduce this theorem from a result of Shimura in [34, Th. 1.5] where a similar but different action: \( f \mapsto f[^x] \) is given. The definition of this action is as follows: For each \( f \in S_{k, \check{ω}, I}(KQ_{ab}) \), the set \( \{ f^σ | σ \in Gal(KQ_{ab}/K) \} \) has only finitely many elements ([34, Prop. 1.3] or else (4.7a) in the text); hence, we can find a positive integer \( N \) so that \( f^σ \in S_{k, \check{ω}, I}(Γ_N; KQ_{ab}) \) for all \( σ \in Gal(Q_{ab}/K) \). This is possible since \( Γ_N \) is the principal congruence subgroup of \( SL_2(τ) \) modulo \( N \) as already remarked. By the strong approximation theorem (cf. [31, Prop. 3.4]), for each \( x \in \mathcal{G}_+(K) \), we can decompose \( x = uα \) for \( u \in U_N \) and \( α \in GL_2^+(𝔽) \). Then we define

\[ f[^x] = f^{p(x)}|_{k, \check{ω} α}.\]

Our action is defined by

\[ f^x = f^{p(x)}|_{k, \check{ω} α} = \det(α)^{\check{ω}} f[^x].\]

Since as already remarked, \( x \mapsto \det(α)^{\check{ω}} \) gives a continuous quasi-character of \( \mathcal{G}_+ \) into \( Φ(τ) \), the action \( f \mapsto f^x \) is well defined. The verification of the properties (4.8b, c) is left to the reader.

For each open compact subgroup \( U \) of \( GL_2(𝔽_f) \), we put

\[ D_U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in U \mid t \in \hat{ℤ}^× \right\}, \quad U(K) = UG_∞^+ \cap \mathcal{G}_+(K),\]
\[ D_{U(K)} = D_U \cap U(K), \quad Γ_U = UG_∞^+ \cap GL_2(𝔽).\]

**Corollary 4.5.** Let \( U \) be an open compact subgroup of \( GL_2(𝔽_f) \), and suppose that

(i) \( ρ: D_U G_∞^+ \to Gal(Q_{ab}/ℚ) \) is surjective,

(ii) \( D_U Γ_U G_∞^+ \) is dense in \( UG_∞^+ \cap \mathcal{G}_+.\)

Then we have that \( S_{k, \check{ω}, I}(Γ_U; K) = H^0(U(K), S_{k, \check{ω}, I}(KQ_{ab})) \) and thus,
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$S_{k, \hat{\omega}, l}(\Gamma_U; K) \cong S_{k, \hat{\omega}, l}(\Gamma_U; \Phi(v)) \otimes_{\Phi(v)} K$. Especially, for $t_i$ as in (4.3), $t_i V_E(N) t_i^{-1}$ and $t_i V_A(N) t_i^{-1}$ satisfy conditions (i) and (ii).

Proof. The last assertion is obvious from the definitions of $t_i$, $V_i(N)$ and $V_E(N)$, and the second assertion follows from the first and (4.7a); so, we prove the first assertion. Let $\hat{\Gamma}_U$ be the closure of $\Gamma_U G_{\infty}$ in $G_+$. Then, by the continuity of the action of Theorem 4.4 and (4.8b), we have

\[ S_{k, \hat{\omega}, l}(\Gamma_U; KQ_{ab}) = H(\hat{\Gamma}_U, S_{k, \hat{\omega}, l}(KQ_{ab})). \]

Since $U(K) = \hat{\Gamma}_U D_{U(K)}$ by (i), (4.8c) shows the result.

For simplicity, we shall write for a moment $U$, $V$, $V_i$ and $V_1$ for $U(N)$, $V_\infty$, $V_1(N) G_{\infty}$ and $V_1 \cap GL_2(F)$, respectively. We shall decompose for $x \in \Delta_1(N)$, $U x U = \bigsqcup U x_i U$; i.e., $V x^{-1} V = \bigsqcup V x_i t_i$. Then we find an index $i$, $\gamma_i$ and $\gamma$ in $GL_2(F)$ such that $t_i x_i = \gamma_i t_i V G_{\infty}$ and $t_i x_i = \gamma_i t_i V G_{\infty}$. The index $i$ is determined independently of $l$ and the correspondence: $i \leftrightarrow j$ is bijective.

Lemma 4.6. Under the above notation, put $V_i(K) = V_i \cap G_+(K)$. Then we have a disjoint decomposition:

\[ V_i(K) \gamma^{-1} V_i(K) = \bigsqcup V_i(K) \gamma_i^{-1} \text{ if } x \in \Delta_1(N). \]

Proof. We have that $V x^{-1} V = V t_i^{-1} \gamma^{-1} t_j V$ and thus $V_i \gamma^{-1} V = V t_i x^{-1} t_j^{-1} V$ and $V x_i t_i^{-1} V = V t_i^{-1} \gamma_i^{-1} V$. This shows that $V_i \gamma^{-1} V = \bigsqcup V_i \gamma_i^{-1}$. Thus $\gamma_i^{-1}$ can be written as $v \gamma^{-1} v'$ with $v \in V_i$ and $v' \in V_j$. By the strong approximation theorem, we may suppose that $v \in SL_2(F_f)$ and therefore $v' \in V_i(K)$. In fact, since $\text{det}(V_i \cap \gamma^{-1} V_i \gamma) = \text{det}(V_i)$, we can find $w \in V_i \cap \gamma^{-1} V_i \gamma$ so that $\text{det}(w) = \text{det}(v)$. Then $v \gamma^{-1} v' = vw^{-1} \gamma^{-1} \gamma w \gamma^{-1} v'$ and $vw \gamma^{-1} v' \in V_j$, $uw^{-1} \in V_i$. By replacing $v$ by $vw$, we may thus assume that $v \in SL_2(F_f)$. This shows that

\[ V_i(K) \gamma^{-1} V_i(K) \supset \bigsqcup V_i(K) \gamma_i^{-1}. \]

We shall show the other inclusion:

\[ V_i(K) \gamma^{-1} V_i(K) \subset \bigsqcup V_i(K) \gamma_i^{-1}. \]

For each $v \gamma^{-1} v' \in V_i(K) \gamma^{-1} V_i(K)$, we can find $v'' \in V_i$ such that $v \gamma^{-1} v' = v'' \gamma_i^{-1}$ for some $l$. By taking the determinants, we learn that $\text{det}(v'') \in \text{det}(v'') F x F_{\infty}$ and $\rho(v'')$ fixes $K \cap Q_{ab}$; hence $v'' \in V_i(K)$. This shows the assertion.

Lemma 4.7. If $x \in \Delta_1(N)$, then $\text{det}(V_i(K) \cap \gamma V_i(K) \gamma^{-1}) = \text{det}(V_i(K))$. Moreover,

\[ V_i(K) \gamma^{-1} V_i(K) = V_i(K) \gamma^{-1} \Gamma_j \Gamma_i \gamma^{-1} \Gamma_j^{-1} = \bigsqcup \Gamma_i \gamma^{-1} \Gamma_j^{-1}. \]
Proof. The last assertion follows from the first by [31, (2.19.3)]; so, we shall prove the first assertion. From the definition of $\Delta_i(N)$, we see easily that 
$$\det(V \cap x^tV^{-t}) = \det(V) \cdot t_i.$$
This shows that 
$$\det(V \cap V_i \gamma^{-1}) = \det(V_i).$$

We shall show that 
$$\left(V_i(K) \cap V_i \gamma^{-1} \cap \mathcal{G}_+(K)\right).$$
In fact, if $u = \gamma u' \gamma^{-1}$ for $u \in V_i(K)$ and $u' \in V_i$, then $u' = \gamma^{-1} u \gamma$ is contained in $V_i(K)$. This shows the above inclusion. The other inclusion is obvious, and hence, we have 
$$V_i(K) \cap V_i \gamma^{-1} \gamma^{-1} = V_i \cap V_i \gamma^{-1} \cap \mathcal{G}_+(K).$$
Then, we see that 
$$\det(V_i(K) \cap V_i \gamma^{-1} \gamma^{-1}) = \det(V_i \cap V_i \gamma^{-1} \gamma^{-1} \cap \mathcal{G}_+(K))$$
$$= \det(V_i \cap V_i \gamma^{-1}) \cap \det(\mathcal{G}_+(K))$$
$$= \det(V_i) \cap \det(\mathcal{G}_+(K)) = \det(V_i(K)).$$

For any field extension $K/\Phi(v)$ (not necessarily inside $\mathbb{C}$), let 
$$S_{k, \omega, l}(\Gamma_E(N); \Phi(v)) \otimes_{\Phi(v)} K \subset K[[q]].$$
For any $\langle v \rangle$-subalgebra $A$ of $K$,
$$S_{k, \omega, l}(\Gamma_E(N); A) = S_{k, \omega, l}(\Gamma_E(N); K) \cap A[[q]].$$
For an $\langle v \rangle$-subalgebra $A$ of $K$ satisfying (3.1) for the given $v$ in $\mathbb{Z}[I]$, we choose the map: $a \mapsto \{a^v\} \in A$, and put 
$$S_{k, \omega, l}(N; A) = \bigoplus_{i=1}^{h} \mathfrak{c}_{v, i} S_{k, \omega, l}(\Gamma_E(N); A),$$
where $c_{v, i}$ is the constant defined in Corollary 4.3. We can naturally identify $S_{k, \omega, l}(N; K)$ with $S_{k, \omega, l}(N; \Phi(v)) \otimes_{\Phi(v)} K$ by Corollary 4.5 and $S_{k, \omega, l}(N; A)$ with an $A$-submodule of $S_{k, \omega, l}(N; K)$.

Theorem 4.8. If $K/\Phi(v)$ is a field extension inside $\mathbb{C}$, then $S_{k, \omega, l}(N; K)$ is stable under the Hecke operator $[V_i(N)x^{-t}V_i(N)]$ as in Section 2 for all $x \in \Delta_i(N)$. Especially, $S_{k, \omega, l}(N; K)$ is stable under $T(n)$ and $T_0(n)$ for all $n$.

Proof. With the notation of Lemma 4.6, we have by (2.6c) that 
$$\left(f| [V_i(N)x^{-t}V_i(N)] \right)_j = \sum_{l} f_l |_{k, \omega} \gamma^{-1}$$
$$= \sum_{l} (f_l \gamma^{-1})$$
Since $V_i(K) \gamma^{-1} V_i(K) = \mathbb{Q} V_i(K) \gamma^{-1}$ by Lemma 4.6, we know from Theorem
4.4 and Corollary 4.5 that if \( f \in H^0(V^i(K), S_k, \omega, i(K_{Qab})) = S_k, \omega, i(\Gamma^i_k(N); K) \), then

\[
\sum_i (f_i)^{n-1} \text{ is contained in } H^0\left(V^i(K), S_k, \omega, i(K_{Qab})\right)
\]

which coincides with \( S_k, \omega, i(\Gamma^i_k(N); K) \) by Corollary 4.5. This shows the assertion.

By this theorem, we can naturally extend the action of the Hecke operators \( T(n), T_0(n) \) and \( T(n, n) \) to \( S^*_k, w, i(N; K) \) for any field extension \( K/\Phi(v) \) (not necessarily in \( \mathbb{C} \)). The extended operators \( T_0(n) \) and \( T(n, n) \) satisfy the same formula as (4.2).

Now we shall generalize Theorem 4.8 to \( \Lambda \)-integral cusp forms. We suppose that \( A \) is a flat \( \mathfrak{a}(v) \)-algebra satisfying (3.1). Firstly we recall the definition of the Hilbert-Blumenthal abelian varieties over \( F \), which will be abbreviated as HBAV. The details of what follows can be found in [27, §1]. An HBAV \( X \) over a base scheme \( S \) is an abelian scheme (in the sense of [25]) over \( S \) with an algebra homomorphism \( m: \mathfrak{a} \rightarrow \text{End}(X/S) \) making \( \text{Lie}(X/S) \) into a locally free sheaf of rank 1 over \( \mathfrak{a} \otimes \mathcal{O}_S \), where \( \mathcal{O}_S \) denotes the structure sheaf of \( S \). Then \( m \) is injective, and the relative dimension of \( X \) over \( S \) is equal to the absolute degree of \( F \). Let \( X^* = \text{Pic}^0(X/S) \) and define \( m^*(a) \in \text{End}(X^*/S) \) by the adjoint of \( m(a) \) for \( a \in \mathfrak{a} \). Then, \( (X^*, m^*) \) becomes naturally an HBAV. The polarization module of \( X \) is by definition

\[
\mathcal{P}(X) = \{ \lambda \in \text{Hom}_S(X, X^*) | \lambda = \lambda^*, m^*(a) \circ \lambda = \lambda \circ m(a) \text{ for all } a \in \mathfrak{a} \}
\]

where "**" indicates the adjoint. Then "\( T \rightarrow \mathcal{P}(X \times_S T/T) \)" is known to be a locally constant sheaf on the étale site over \( S \), which has values in the category of projective \( \mathfrak{a} \)-modules of rank 1 ([27, Prop. 1.17]). Let \( \mu: X \times_S X \rightarrow X \) be the multiplication morphism on \( X \) and \( p_1, p_2: X \times_S X \rightarrow X \) be the two projections. Consider the following sheaf on \( X \times_S X \) made out of each invertible sheaf \( L \) on \( X \):

\[
\psi(L) = \mu^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}.
\]

Note that the value at \( X \) of the relative Picard functor for \( X/S \) is given by

\[
\text{Pic}_{X/S}(X) = \left\{ \text{group of invertible sheaves on } X \times_S X \right\} \big/ \left\{ \text{subgroup of sheaves of the form } p_2^*(M) \text{ for an invertible sheaf } M/X \right\}
\]

Then \( \psi(L) \) gives an \( X \)-valued point of \( \text{Pic}_{X/S}(X) \). Now suppose that \( L \) is relatively ample. Then, it is known that \( \psi(L) \) is contained in the connected component \( \text{Pic}^{\leq}_X(X) \) ([25, VI.2]). Since \( \text{Pic}^{\leq}_X(X) \cong \text{Hom}_S(X, X^*) \)
we have a homomorphism corresponding to $\psi(L)$:

$$\Lambda(L): X \to X^* \quad \text{over } S.$$ 

By definition, a polarization of $X$ is an $S$-homomorphism $\lambda: X \to X^*$ such that for every geometric point $s$ of $S$, the induced $\lambda: \tilde{X} \to \tilde{X}^*$ for the fibers $\tilde{X}$ and $\tilde{X}^*$ at $s \in S$ is of the form $\Lambda(\tilde{L})$ for some ample invertible sheaf $\tilde{L}$ on $\tilde{X}$ ([25, I.2.6.3]). Put

$$\mathcal{P}_+(X) = \{ \lambda \in \mathcal{P}(X) | \lambda \text{ is a polarization for } X \}.$$ 

Then, it is well-known that there is an $\otimes$-linear embedding $i: \mathcal{P}(X) \to F$ such that $i$ induces an isomorphism: $\mathcal{P}_+(X) \cong i(\mathcal{P}(X)) \cap F^+$. Thus $\mathcal{P}_+(X)$ gives a notion of positivity on $\mathcal{P}(X)$ ([27, 1.15] [3, §4]).

In order to give the definition of the spaces of modular forms in this context, we fix a positive integer $N_0$ and a fractional ideal $c$. Let $d'$ be the absolute different of $F$. We shall consider the moduli problem of quadruples $(X, \lambda, \omega, i)$, where $X$ is an HBAV over a ring $A$, $\lambda: \mathcal{P}(X) \cong c$ which induces an isomorphism: $\mathcal{P}_+(X) \cong c_+$, $\omega$ is a base of $\otimes Z A$-module $H^0(X, \Omega^1_{X/A})$ and $i: N_0^{-1}d^{-1}d^{-1} \otimes Z \mu_{N_0} \hookrightarrow X$ is a $\Gamma_00(N_0)$-structure over $A$ (cf. [3, §§]). Here $\mu_{N_0}$ is a finite flat group scheme over $Z$ which is the kernel of the multiplication by $N_0$ on $G_m$. To give $\omega$ is equivalent to giving an $\otimes$-linear isomorphism: $\text{Lie}(X/A) \cong d' \otimes Z A$. Let $k, v \in Z[1]$ be as in Section 3, and suppose that $A$ is an $\otimes$-algebra. Then the modular forms for $\Gamma_00(N_0)$ of weight $k$ over $A$ are functions $f$ of isomorphism classes of quadruples $(X, \lambda, \omega, i)_A$ for all $A$-algebras $A'$ such that

\begin{align*}
(4.9a) & \quad f((X, \lambda, \omega, i)_{A'}) \in A', \\
(4.9b) & \quad f((X, \lambda, a\omega, i)_{A'}) = a^{-k}f((X, \lambda, \omega, i)_{A'}) \\
& \quad \text{for } a \in (\otimes Z A')^\times, \\
(4.9c) & \quad \text{If } \rho: A' \to A'' \text{ is a homomorphism of } A\text{-algebras}, \\
& \quad f((X, \lambda, \omega, i)_{A'} \times A'') = \rho(f((X, \lambda, \omega, i)_{A'})).
\end{align*}

When $F = \mathbb{Q}$, an additional condition on the holomorphy at each cusp is necessary. However, the case of $F = \mathbb{Q}$ has already been treated in [12, §1]; so, we hereafter assume that $F \neq \mathbb{Q}$. The space of modular forms over $A$ will be denoted by $\mathcal{M}_k(\Gamma_00(N_0), c; A)$. 

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An (unramified) cusp on $\Gamma_0(N_0)$ over $A$ is given by the following data ([3, §5], [19, 1.1]):

- (4.10a) two ideals $a, \ell$ with $a\ell^{-1} = e$,
- (4.10b) an $\imath$-linear isomorphism $\varepsilon: N_0^{-1}x/\imath \cong N_0^{-1}a^{-1}/a^{-1}$,
- (4.10c) an $\imath \otimes \mathbb{Z}$ $A$-linear isomorphism $j: a^{-1} \otimes \mathbb{Z} A \cong \imath \otimes \mathbb{Z} A$.

The evaluation of modular forms at the generalized Tate curve: $(\text{Tate}_{\imath, \ell}(q), \lambda_{\text{can}}, \omega_{\text{can}}(j), i_{\text{can}}(\varepsilon))$ as in [19, (1.1.13–17)] (or [27, §4]) gives the following $q$-expansion of $f \in \mathcal{M}_k(\Gamma_0(N_0), \varepsilon; A)$:

$$f(a, \ell, \varepsilon, j) = \sum_{\xi \in a\ell^{-1} \cup \{0\}} a(\xi; a, \ell, \varepsilon, j; f)q^{\xi} \in A[[q]]_{a\ell}.$$

If $\mathcal{N}_{F/\mathbb{Q}}(a)$ is a unit in $A$, then we have a canonical isomorphism

$$j_{\text{can}}: a^{-1} \otimes \mathbb{Z} A \cong \imath \otimes \mathbb{Z} A.$$

For each $a \in F_A^\times$ with $a = a\imath$, we can choose an isomorphism

$$\varepsilon_a: N_0^{-1}x/\imath = N_0^{-1}x/\imath = N_0^{-1}a^{-1}/a^{-1} \cong N_0^{-1}a^{-1}/a^{-1} = a^{-1}x.$$

When $\mathcal{N}_{F/\mathbb{Q}}(a)$ is a unit in $A$, we write $f(a)$ for the $q$-expansion of $f$ in $\mathcal{M}_k(\Gamma_0(N_0), \varepsilon; A)$ at $(a\imath, a\imath^{-1}, \varepsilon_a, j_{\text{can}})$. Here are the $q$-expansion principles (cf. [19, (1.2.15–16)], [3, (5.4–5)]):

- (4.12a) If $f(a, \ell, \varepsilon, j) = 0$ for $f \in \mathcal{M}_k(\Gamma_0(N_0), \varepsilon; A)$ at a cusp $(a, \ell, \varepsilon, j)$, then $f = 0$;
- (4.12b) For each $\imath(v)$-subalgebra $A_0$ of $A$, if $(a, \ell, \varepsilon, j)$ is in fact a cusp over $A_0$ and if $f(a, \ell, \varepsilon, j) \in A_0[[q]]_{a\ell}$ for $f \in \mathcal{M}_k(\Gamma_0(N_0), \varepsilon; A)$, then $f \in \mathcal{M}_k(\Gamma_0(N_0), \varepsilon; A_0)$.

Now we shall clarify the relation between the classical space $S_k(N, \lambda, \omega, i)(\Gamma_F^I(N); A)$ and newly defined $\mathcal{M}_k(\Gamma_0(N), \varepsilon; A)$. To do this, for each integral ideal $a$, put

$$\Gamma_0(N_0; a) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(F) \mid a, d \in \imath, c \in N_0a, a - 1, d - 1 \in N_0\imath, b \in a^{-1} \right\}.$$

Over the complex number field $\mathbb{C}$, to give a quadruple $(X, \lambda, \omega, i)$ is equivalent to giving a triple $(\mathcal{L}, \lambda, i)$ consisting of an $\imath$-lattice $\mathcal{L}$ in $F \otimes \mathbb{C} = F_{\mathbb{C}} (\cong \mathbb{C}^I)$, an $\imath$-linear embedding $i: N_0^{-1}\imath^{-1}/\imath^{-1} \to N_0^{-1}\mathbb{L}/\mathbb{L}$ and an isomor-
phism $\lambda: \mathcal{L} \wedge \mathcal{L} \cong d^{-1}c^{-1}$ which is induced by an alternating form $\langle \cdot, \cdot \rangle: \mathcal{L} \times \mathcal{L} \to d^{-1}c^{-1}$ such that

$$\langle x, y \rangle = a \Im(xy)$$

for some $a \in F^\times_+$. (Here $\Im(xy)$ means the imaginary part of $xy \in F_C$ over $F_{\infty} = F \otimes_{\mathbb{Q}} \mathbb{R}$.) In fact, as for the differential $\omega$ on $X(C) = F_C/\mathcal{L}$, we take the usual one induced by $du = \sum_{a \in I} du_a$ by identifying $F_C$ with $\mathbb{C}$. The alternating form $\lambda$ gives the polarization of the complex torus $X(C)$ which turns $X(C)$ into an abelian variety over $C$. Thus we have the quadruple $(X, \lambda, \omega, i)/C$ out of $(\mathcal{L}, \lambda, i)$ which exhausts all the isomorphism classes of $(X, \lambda, \omega, i)$ over $C$. Then, by (4.9b), we have

$$f(a, (a\tilde{a})^{-1}\lambda, ai) = a^{-k}f(\mathcal{L}, \lambda, i) \quad \text{for } a \in F_C^\times ([3, (5.6)]).$$

For each $z \in H^1 \subset F_C$, we consider a lattice $\mathcal{L}_z = 2\pi i(d^{-1}a^{-1} + bz)$, where we regard $a$ and $b$ as $\mathfrak{s}$-submodules of $F_C$ naturally. Define $\lambda_{\text{can}}: \mathcal{L}_z \times \mathcal{L}_z \to d^{-1}c^{-1}$ by $\lambda_{\text{can}}((2\pi i)(a + bz), (2\pi i)(c + dz)) = ad - bc$ and

$$i_{\text{can}}: N_0^{-1}d^{-1}/d^{-1} \to X_z = F_C/\mathcal{L}_z$$

by the composition:

$$\left(N_0^{-1}d^{-1}/d^{-1} \right) = \left(N_0^{-1}d^{-1}/d^{-1} \right) \otimes \mathbb{C} \otimes d^{-1} \otimes \text{id} = \left(N_0^{-1}d^{-1}/d^{-1} \right) \otimes \mathbb{C} \otimes d^{-1} \otimes \text{id} = N_0^{-1}d^{-1}/d^{-1} \otimes N_0^{-1} \mathcal{L}_z/\mathcal{L}_z \subset X_z.$$

The linear fractional transformation $z \mapsto \gamma(z)$ induces an isomorphism:

$$\left(j_l(\gamma, z) \mathcal{L}_{\gamma(z)}, (j_l(\gamma, z))^{-1}\lambda_{\text{can}}, j_l(\gamma, z) i_{\text{can}} \right) \cong \left(\mathcal{L}_z, \lambda_{\text{can}}, i_{\text{can}} \right)$$

if and only if $\gamma \in \Gamma_0(0; a^2c^{-1}d)$. Therefore the function on $H^1$

$$f_{a, \ell, \epsilon}(z) = f(\mathcal{L}_z, \lambda_{\text{can}}, i_{\text{can}}) \quad \text{for } f \in M_k(\Gamma_0(N_0), \epsilon; C)$$

gives a classical modular form on $\Gamma = \Gamma_0(N_0; a^2c^{-1}d)$ satisfying

(i) $f_{a, \ell, \epsilon}: H^1 \to C$ is holomorphic,

(ii) $f_{a, \ell, \epsilon} | \gamma = f_{a, \ell, \epsilon}$ for $\gamma \in \Gamma$.

We denote by $M_k(\Gamma_0(N_0; a))$ the space of functions satisfying the above two conditions for $\Gamma = \Gamma_0(N_0; a)$. Moreover, we have:

(4.13a) The correspondence: $f \mapsto f_{a, \ell, \epsilon}$ gives an isomorphism:

$$M_k(\Gamma_0(N_0), \epsilon; C) \cong M_k(\Gamma_0(0; a^2c^{-1}d))$$

such that the Fourier expansion of $f_{a, \ell, \epsilon}$ at $\infty$ gives the $q$-expansion of $f$ at $(a, \ell, \epsilon, j_{\text{can}})$ for $a \in F_C^\times$ with $a = a \ell$ (by replacing $e_F(\xi z)$ by $q^\ell$).
This can be strengthened as follows: Let \( A \) be an \( \mathfrak{v}(v) \)-subalgebra of \( \mathcal{C} \) and suppose that \( \mathcal{N}_{F/Q}(a) \) is invertible in \( A \). Then for any \( A \)-algebra \( A' \) in \( \mathcal{C} \),

\[(4.13b) \text{The correspondence: } f \mapsto f_{a, e, \mathfrak{v}} \text{ induces a } q\text{-expansion preserving isomorphism:}
\]

\[
\mathcal{M}_k(\Gamma_{00}(N_0), e; A') \cong \mathcal{M}_k(\Gamma_{00}(N_0; a^2c^{-1}d); A')
\]

\[
= \mathcal{M}_k(\Gamma_{00}(N_0; a^2c^{-1}d)) \cap A'[\lfloor q \rfloor]_{a\mathfrak{v}}.
\]

Let \( \Gamma = \Gamma_{00}(N_0; e) \). By the strong approximation theorem, if we write \( \hat{\Gamma} \) for the closure of \( \Gamma \) in \( SL_2(F) \), then we know that \( SL_2(F) = SL_2(F) \cdot \hat{\Gamma} = \hat{\Gamma} \cdot SL_2(F) \). Thus for each \( x \in SL_2(F) \), we may choose \( u \in \hat{\Gamma} \) and \( \alpha \in SL_2(F) \) such that \( x = u\alpha \). For \( f \) in \( \mathcal{M}_k(\Gamma) \), put \( f|_k x = f|_{k, w} \alpha \). This is independent of the choice of \( \alpha \) and \( w \) since \( \det(\alpha) = 1 \) and is determined only by \( x \).

We shall quote a theorem of Deligne and Ribet [3, 5.8]:

**Theorem 4.9.** Suppose that \( A \) is a discrete valuation ring of a finite extension of \( \Phi(v) \). Write \( a = ax \) for \( a \in F^\times \), and suppose that \( \mathcal{N}_{F/Q}(a) \) is invertible in \( A \). If \( f \in \mathcal{M}_k(\Gamma_{00}(N_0), e; A') \) for an \( A \)-algebra \( A' \) inside \( \mathcal{C} \), then

\[
f(1)|_{k}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = f(a) \quad \text{in } A'[\lfloor q \rfloor]_{(a^2c^{-1})}.
\]

Note that \( f(1) \in \mathcal{M}_k(\Gamma_{00}(N_0; c^{-1}d)) \) and \( f(a) \in \mathcal{M}_k(\Gamma_{00}(N_0; a^2c^{-1}d)) \).

**Theorem 4.10.** For any extension \( K/\Phi(v) \), if an \( \mathfrak{v}(v) \)-subalgebra \( A \) of \( K \) satisfies \((3.1)\), we have a natural isomorphism: \( S_{k, \hat{\mathfrak{w}}, \mathfrak{v}}(N; K) \cong S_{k, \hat{\mathfrak{w}}, \mathfrak{v}}(N; A) \otimes_A K \).

This result follows from [32, Th. 1] and [33, p. 683] when \( k \in Z \cdot t \) (i.e., \( v = 0 \)) in view of Corollary 4.5. The general case can be derived from a result of Rapoport [27] (see also [3, p. 258]) by algebra-geometric means. We shall give a proof of this fact in Section 7 by cohomological means.

**Theorem 4.11.** Let \( A \) be an integrally closed domain containing \( \mathfrak{v}(v) \). Suppose that \( A \) is finite flat over either of \( \mathfrak{v}(v) \) or \( Z_p \) and satisfies \( (3.1) \) for \( v \in Z[1] \). Then \( S_{k, \hat{w}, \mathfrak{v}}(N; A) \) is stable under \( T_0(n) \) for all integral ideals \( n \).

**Proof.** By Corollary 4.2, what we have to show is the stability of \( S_{k, \hat{w}, \mathfrak{v}}(N; A) \) under \( \mathcal{N}_{F/Q}(\mathfrak{e})T_0(\mathfrak{e}, \mathfrak{e}) \) for all ideals \( \mathfrak{e} \) prime to \( N \). We fix a map: \( a \mapsto \{ a^v \} \in A \) as in Section 3 and define \( T_0(\mathfrak{e}, \mathfrak{e}) \) with respect to this map. For each prime ideal \( \mathfrak{p} \) of \( A \), let \( A_{\mathfrak{p}} \) be the localization of \( A \) at \( \mathfrak{p} \). Then \( A_{\mathfrak{p}} \) is a valuation ring and \( A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}} \) in the quotient field \( K \) of \( A \). We now by
definition that

\[ S_{k, \tilde{w}, t}(N; A) = \bigcap_{\chi} S_{k, \tilde{w}, t}(N; A_{\chi}) \quad \text{inside } S_{k, \tilde{w}, t}(N; K). \]

Then the stability of \( S_{k, \tilde{w}, t}(N; A) \) under \( \mathcal{N}_{F/K}(\ell)T_{0}(\ell, \ell) \) follows from that of \( S_{k, \tilde{w}, t}(N; A_{\chi}) \). Thus we may assume that \( A \) is a discrete valuation ring of residual characteristic \( \ell \). If \( A \) is finite flat over \( \mathbb{Z}_{\ell} \), then we can find a finite extension \( K_{0}/\Phi(v) \) inside \( C \) such that the quotient field \( K \) of \( A \) is the closure of \( K_{0} \) in \( \overline{Q}_{\ell} \). Then

\[ S_{k, \tilde{w}, t}(N; A) = \bigoplus_{\ell} c_{\nu, t} S_{k, \tilde{w}, t}(\Gamma_{E}^{\nu}(N); A) \]

and

\[ S_{k, \tilde{w}, t}(\Gamma_{E}^{\nu}(N); A) = S_{k, \tilde{w}, t}(\Gamma_{E}^{\nu}(N); K) \cap A[[q]]_{i}. \]

We put, for each positive integer \( N_{0} \),

\[ \mathcal{M}_{k}(\Gamma_{00}(N_{0}; \alpha_{i}^{-1}); K) = \mathcal{M}_{k}(\Gamma_{00}(N_{0}; \alpha_{i}^{-1}); K_{0}) \otimes_{K_{0}} K \quad \text{and} \]

\[ \mathcal{M}_{k}(\Gamma_{00}(N_{0}; \alpha_{i}^{-1}); A) = \mathcal{M}_{k}(\Gamma_{00}(N_{0}; \alpha_{i}^{-1}); K) \cap A[[q]]_{i}. \]

Now we take a positive integer \( N_{0} \) contained in the ideal \( N \). By taking \( c = \alpha_{i}^{\nu} \), we have the inclusion:

\[ S_{k, \tilde{w}, t}(\Gamma_{E}^{\nu}(N); A) \hookrightarrow \mathcal{M}_{k}(\Gamma_{00}(N_{0}; \alpha_{i}^{-1}); A) = \mathcal{M}_{k}(\Gamma_{00}(N_{0}), c; A). \]

(4.14)

\[ f(1) = f_{s, a_{i}^{\nu}, \epsilon_{i}} \rightarrow f \]

Note that, for \( a = a_{i}^{\nu} \) \((a \in F_{f}^{\times}) \) with \((a, N_{0}l) = 1\),

\[ f|_{T}(a, a)(x) = f(xa) = f\left( x\begin{pmatrix} a^{2} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 1 & a \end{pmatrix} \right) \quad (f \in S_{k, \tilde{w}, t}(N; C)). \]

Write \( s = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \) and \( r = \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix} \). Then \( f|_{T}(a, a) = f|_{k, \tilde{w}}sr \). Define

\[ V(M) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in V_{1}(M) \mid b \in M \right\} \quad \text{for each ideal } M \text{ of } \tilde{z}. \]

For the above \( s \) and \( r \), we can find a sufficiently small ideal \( M \) so that

\[ sV(M)s^{-1} \subset V_{1}(N_{0}) \quad \text{and} \quad rV(M)r^{-1} \subset V_{1}(N_{0}). \]

We can also decompose by using the same \( t_{i} \) as in (4.3)

\[ G_{A} = \prod_{i=1}^{h} \text{GL}_{2}(F) t_{i} V(M) G_{\infty}. \]

Note that \( \det(s) = 1 \). Then we can write \( t_{1}sr = \gamma t_{1}u_{1}, \alpha t_{1}u_{2} = t_{1}s \) and \( \beta t_{1}u_{3} = \]
$t_j r$ for $u_1, u_2, u_3 \in V(M)$ and $\alpha, \beta, \gamma \in \text{GL}_2^+(F)$. Then we have that

$$t_j s r = \alpha t_j u_2 r = \alpha t_j r^{-1} u_2 r = \alpha \beta t_j u_3 r^{-1} u_2 r.$$ 

Since $u_3 r^{-1} u_2 r \in V(N_0)$, by writing $f_i$ for $f_i'$, we see from (2.6c) that

$$\left( f \mid_{k, \tilde{\omega}} s r \right)_i = f_j \mid_{k, \tilde{\omega}} \gamma = \left( f_j \mid_{k, \tilde{\omega}} \alpha \right)_i \mid_{k, \tilde{\omega}} \beta.$$ 

We may assume that $s \equiv \alpha \mod{M M_2(\tilde{\epsilon})}$ and $\det(\alpha) = 1$ by the strong approximation theorem, since $\det(s) = 1$. Then, we see that

$$f_j \mid_{k, \tilde{\omega}} \alpha = f_j \mid_{k} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = f_j(a) \text{ by Theorem 4.9.}$$

On the other hand, we can write $\beta_0 a_i u_0 = a_j a^{-2}$ for $\beta_0 \in F^\times$ and $u_0 \in \tilde{\epsilon} X F^\times$. Then, we can choose $\begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix}$ as $\beta$. Thus we see that if $f_j(a) = \sum_{\xi} a(\xi) q^{\xi}$,

$$\left( f \mid T(a, a) \right)_i = \beta_0^k \sum_{\xi} a(\xi) q^{\kappa \xi}.$$ 

If $f \in S^*_{k, \omega, i}(N; A)$, then $f_j \in c_{v, j} \mathcal{M}_i(\Gamma_{00}(N_0), a_j d'; A)$ by the inclusion (4.14). Then by Theorem 4.9, we know that

$$\left( f \mid T(a, a) \right)_i \in \beta_0^k \sum_{\xi} a(\xi) q^{\kappa \xi}.$$ 

Note that $\beta_0 a_i = a_j a^{-2}$ and $c_{v, j} = \mathcal{N}_{F/Q}(a_j)^{-1}(a_j d')$. Since $\mathcal{N}_{F/Q}(a)$ is invertible in $A$ (i.e. $a$ is prime to $\ell$), we know from a straightforward calculation that

$$\mathcal{N}_{F/Q}(a) \{ a^{-2v} \} \beta_0^k c_{v, j} A = c_{v, i} A.$$ 

This shows that if $a$ is prime to $\ell N_0$, then $S^*_{k, \omega, i}(N; A)$ is stable under $\mathcal{N}_{F/Q}(a) T_0(a, a)$. Now write simply $m$ for $[t + n + 2v]$. Since we know from (3.9) that

$$\mathcal{N}_{F/Q}(a) T_0(a, a) = \mathcal{N}_{F/Q}(a) \{ a^{-2v} \} \langle a \rangle_n,$$

and $\langle a \rangle_n$ only depends on the class of $a$ in $\text{Cl}_F(N)$ as shown in Section 3, we can take ideals $\alpha, \beta$ of $\pi$ such that $a \alpha \beta$ is prime to $N_0 \ell$, $\alpha$ and $\beta$ are in the same class in $\text{Cl}_F(N)$ and $\mathcal{N}_{F/Q}(a) \{ a^{-2v} \} A + \mathcal{N}_{F/Q}(\beta) \{ \beta^{-2v} \} A = A$. Then $T_0(a, a) + T_0(\ell', \varepsilon)$ is a unit multiple of $\langle a \rangle_n$. This shows that under the action of $\text{Cl}_F(N)$, $S^*_{k, \omega, i}(N; A)$ is stable. For a general ideal $a$ prime to $N$ but not necessarily prime to $\ell N_0$, we know that

$$\mathcal{N}_{F/Q}(a) T_0(a, a) = \mathcal{N}_{F/Q}(a) \{ a^{-2v} \} \langle a \rangle_n.$$ 

Since $\mathcal{N}_{F/Q}(a) \{ a^{-2v} \}$ is an element of $A$ (even when $-t < n \leq 0$; i.e., $0 < k \leq 2t$), $S^*_{k, \omega, i}(N; A)$ is stable under $\mathcal{N}_{F/Q}(a) T_0(a, a)$, which finishes the proof.
Here we record a byproduct of the proof of Theorem 4.11.

**Corollary 4.12.** Let $A$ be an $\mathfrak{t}(v)$-algebra as in Theorem 4.11. Then $S_{k, w, I}(N; A)$ is stable under the action of the finite group $\text{Cl}_F(N)$ via the operator $\langle a \rangle_n$ for $a \in \mathbb{I}(N)$.

For the stability of $S_{k, w, I}(N; A)$ under $T_0(\mathfrak{n})$, the assumption that $v$ (and $w$) is integral and $n - 2v$ (and $k - 2w$) is absolutely necessary as otherwise one can construct counterexamples [34, Remark 2.9].

5. Duality theorems between Hecke algebras and spaces of cusp forms

Let $A$ be a Dedekind domain containing $\mathfrak{t}(v)$ (inside $C$ or $\overline{Q}_p$) satisfying (3.1). Then by Theorem 4.1, $S_{k, w, I}(N; A)$ is stable under $T_0(\mathfrak{n})$ for all ideals $\mathfrak{n} \subset \mathfrak{t}$ and hence is stable under $\mathcal{H}_{k, w}(N; A)$. We shall define a pairing $\langle h, f \rangle$ on $\mathcal{H}_{k, w}(N; A) \times S_{k, w, I}(N; A)$ to $A$ by

$$\langle h, f \rangle = a(\mathfrak{t}, f|h).$$

**Theorem 5.1.** The pairing (5.1) induces isomorphisms:

$$\text{Hom}_A(\mathcal{H}_{k, w}(N; A), A) \cong S_{k, w, I}(N; M_2(F); A),$$

$$\text{Hom}_A(S_{k, w, I}(N; M_2(F); A), A) \cong \mathcal{H}_{k, w}(N; A).$$

**Proof:** Firstly, we shall assume $A$ to be a field. Since $S_{k, w, I}(N; A)$ and $\mathcal{H}_{k, w}(N; A)$ are of finite dimension by Corollary 4.5 or Theorem 3.1, we shall prove the nondegeneracy of the pairing. Suppose $\langle h, f \rangle = 0$ for all $h$; then from Corollary 4.2,

$$a(\mathfrak{n}, f) = a(\mathfrak{t}, f|T_0(\mathfrak{n})) = \langle T_0(\mathfrak{n}), f \rangle = 0 \quad \text{for all ideals } \mathfrak{n}.$$  

Now $f = 0$ by Proposition 4.1. Conversely if $\langle h, f \rangle = 0$ for all $f$, then for all $\mathfrak{n} \in \mathbb{I}(1),$

$$a(\mathfrak{n}, f|h) = a(\mathfrak{t}, f|T_0(\mathfrak{n})h) = a(\mathfrak{t}, f|T_0(\mathfrak{n})h) = \langle h, f|T_0(\mathfrak{n}) \rangle = 0.$$

This shows the proof in the case where $A$ is a field. For the general case, we may assume that $A$ is a valuation ring by localizing $A$ at prime ideals if necessary. Let $L$ be the quotient field of $A$. What we have to show is the isomorphism:

$$S_{k, w, I}(N; A) \cong \text{Hom}_A(\mathcal{H}_{k, w}(N; A), A).$$

By definition, $\mathcal{H}_{k, w}(N; A)$ is an $A$-subalgebra of $\mathcal{H}_{k, w}(N; L)$. Since $\mathcal{H}_{k, w}(N; A)$ is finite over $A$, $\mathcal{H}_{k, w}(N; A) \otimes_A L$ is a subalgebra of $\mathcal{H}_{k, w}(N; L)$. Thus we can extend any $\phi \in \text{Hom}_A(\mathcal{H}_{k, w}(N; A), A)$ to an $L$-linear map $\phi: \mathcal{H}_{k, w}(N; L) \to L$. 

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Then by the duality for L already proved, we can find \( f \in S_{k, w, l}^*(N; L) \) such that \( \phi(h) = \langle h, f \rangle \) for all \( h \in \mathcal{H}_{k, w}(N; A) \). Then we know that for all ideals \( \mathfrak{n} \in \Pi(1) \),

\[
a(\mathfrak{n}, f) = a(\mathfrak{n}, f|T_0(\mathfrak{n})) = \langle T_0(\mathfrak{n}), f \rangle = \phi(T_0(\mathfrak{n})) \in A,
\]

since \( T_0(\mathfrak{n}) \in \mathcal{H}_{k, w}(N; A) \). This shows that \( f \in S_{k, w, l}^*(N; A) \), and the proof is completed.

In the same manner as in [15, §0], we get from Th. 5.1:

**COROLLARY 5.2.** Let \( C \) denote \( \mathbb{C} \) or \( \overline{\mathbb{Q}}_p \), according as \( A \subset \mathbb{C} \) or \( A \subset \overline{\mathbb{Q}}_p \). Then we have bijections:

\[
\text{Spec}\left( \mathcal{H}_{k, w}(N; A) \right)_{/A}(C) = \text{Hom}_{Aalg}\left( \mathcal{H}_{k, w}(N; A), C \right)
\]

\[
\equiv \{ f \in S_{k, w, l}^*(N; C) | f|T_0(\mathfrak{n}) = a(\mathfrak{n}, f) \text{ for all } \mathfrak{n} \},
\]

\[
\text{Spec}\left( \mathcal{H}_{k, w}^{\text{ord}}(N; A) \right)_{/A}(\overline{\mathbb{Q}}_p) = \text{Hom}_{Aalg}\left( \mathcal{H}_{k, w}^{\text{ord}}(N; A), \overline{\mathbb{Q}}_p \right)
\]

\[
\equiv \{ f \in S_{k, w, l}(N; C) | f|T_0(\mathfrak{n}) = a(\mathfrak{n}, f) \text{ with } |a(p, f)|_p = 1 \}.
\]

Now we fix a finite extension \( K \) of \( \mathbb{Q}_p \) inside \( \overline{\mathbb{Q}}_p \), containing \( \Phi(v) \), and let \( \mathcal{O} \) denote the \( p \)-adic integer ring of \( K \). We fix an ideal \( N \) prime to \( p \), and define, for \( A = \mathcal{O} \) or \( K \)

\[
S_{k, w, l}(Np^\infty; A) = \lim_{\alpha} S_{k, w, l}^*(Np^\alpha; A).
\]

For each element \( f \in S_{k, w, l}^*(Np^\infty; K) \), we shall define a \( p \)-adic norm by

\[
(5.3) \quad |f|_p = \text{Sup}_{\mathfrak{n}} |a(\mathfrak{n}, f)|_p.
\]

By Theorem 4.10, \( |f|_p \) is a well defined real number. Let \( \overline{S}_{k, w, l}(Np^\infty; A) \) for \( A = \mathcal{O} \) or \( K \) be the completion of \( S_{k, w, l}(Np^\infty; A) \) under this norm. By definition, the function

\[
a: \Pi(1) \times S_{k, w, l}(Np^\infty; A) \to A \text{ given by } (a, f) \mapsto a(a, f)
\]

is extended by continuity to \( \Pi(1) \times \overline{S}_{k, w, l}(Np^\infty; A) \), and the norm of each element \( f \) of \( \overline{S}_{k, w, l}(Np^\infty; A) \) is again given by (5.3). By the commutativity of (3.5) and by Th. 4.11, \( \mathcal{H}_{k, w}(Np^\infty; \mathcal{O}) \) acts naturally and faithfully on \( S_{k, w, l}(Np^\infty; \mathcal{O}) \) and by continuity, its action extends to \( \overline{S}_{k, w, l}(Np^\infty; \mathcal{O}) \). Thus we can define a pairing

\[
\langle , \rangle: \mathcal{H}_{k, w}(Np^\infty; \mathcal{O}) \times \overline{S}_{k, w, l}(Np^\infty; \mathcal{O}) \to \mathcal{O} \text{ again by (5.1)}.
\]
Theorem 5.3. The pairing \( \langle , \rangle \) induces isomorphisms:
\[
\kappa_{k,w}(Np^\infty; \mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(\tilde{S}^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}), \mathcal{O}),
\]
\[
\tilde{S}^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(\kappa_{k,w}(\mathcal{N}p^\infty; \mathcal{O}), \mathcal{O}).
\]

One can derive this theorem from Theorem 5.1 in exactly the same manner as in [11, II, Th. 1.3]; so, we omit the proof.

Corollary 5.4. If \( v = 0 \) and \( k \geq 2t \) \( (k \sim 0) \), we have a natural isomorphism:
\[
\tilde{S}^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}) \cong \tilde{S}^{2k}_2, I(\mathcal{N}p^\infty; \mathcal{O})
\]
which preserves the map: \( (a, f) \mapsto a(a, f) \). Moreover, for the ordinary part, we have a similar isomorphism for all pairs \( (k, k') \) with \( k \sim k' \) and \( k \geq k' \geq 2t \):
\[
e^k_{k,w} I(\mathcal{N}p^\infty; \mathcal{O}) \cong e^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}) \quad (w = v + k - t, \ w' = v + k' - t),
\]
which preserves the map: \( (a, f) \mapsto a(a, f) \).

This follows from Theorems 5.3, 3.2 and 3.3. Hereafter identifying \( e^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}) \) for all \( n \geq 0 \) \( (k = n + 2t, \ w = v + k - t = n + v + t) \), we write \( S_v^{\text{ord}}(\mathcal{N}; \mathcal{O}) \) for \( e^k_{k,w}, I(\mathcal{N}p^\infty; \mathcal{O}) \) and put \( S_v^{\text{ord}}(\mathcal{N}; \overline{\mathbb{Q}}_p) = S_v^{\text{ord}}(\mathcal{N}; \mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \).

Corollary 5.5. We have a bijection:
\[
\text{Spec}(\mathfrak{h}^{\text{ord}}_v(\mathcal{N}; \mathcal{O}))_{/\mathcal{O}}(\overline{\mathbb{Q}}_p) = \text{Hom}_{\mathcal{O}, \text{alg}}(\mathfrak{h}^{\text{ord}}_v(\mathcal{N}; \mathcal{O}), \overline{\mathbb{Q}}_p)
\]
\[
\cong \{ f \in S_v^{\text{ord}}(\mathcal{N}; \overline{\mathbb{Q}}_p) | f|T_0(n) = a(n, f) f \}
\]
for all \( n \in \mathfrak{I}(1) \).

This follows from Theorem 5.3. For the proof, see the proof of the next theorem whose assertion is a little stronger than Corollary 3.5. We shall prove this by assuming Theorem 3.4 which will be in turn proved in Section 12.

Theorem 5.6. Let \( \lambda: \mathfrak{h}^{\text{ord}}_v(\mathcal{N}; \mathcal{O}) \to \overline{\mathcal{F}} \) be a \( \Lambda \)-algebra homomorphism, \( \mathcal{X} \) be the quotient field of the image of \( \lambda \) and \( \mathcal{I} \) be the integral closure of \( \lambda \) in \( \mathcal{X} \). For each \( P \in \mathcal{X}(\mathcal{I}) = \text{Hom}_{\mathcal{O}, \text{alg}}(\mathcal{I}, \overline{\mathbb{Q}}_p) \), we define \( \lambda_P: \mathfrak{h}^{\text{ord}}_v(\mathcal{N}; \mathcal{O}) \to \overline{\mathbb{Q}}_p \) by \( P \circ \lambda \). Then there exists a unique \( p \)-adic cusp form \( f_P \in S_v^{\text{ord}}(\mathcal{N}; \overline{\mathbb{Q}}_p) \) such that \( f_P|T_0(n) = \lambda_P(T_0(n)) f_P \) and \( a(n, f_P) = \lambda_P(T_0(n)) \) for all \( n \)-ideals \( n \). If \( P \in \mathcal{X}_{\text{alg}}(\mathcal{I}) \) and \( n(P) \geq 2v \), then \( \lambda_P(T_0(n)) \) is an algebraic number in \( \overline{\mathbb{Q}} \) for all \( n \), and when \( a(n, f_P) = \lambda_P(T_0(n)) \) as a complex number by the fixed embeddings: \( \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \) and \( \overline{\mathbb{Q}} \to \mathbb{C} \), \( f_P \) coincides with a complex cusp form in \( S_k^{\text{alg}}, I(\mathcal{N}p^{a(P)}, e_P; \mathbb{C}) \) for \( k = n(P) - 2v + 2t, \ w = n(P) - v + t \). Conversely, suppose there is a non-zero common eigenform \( f \) in \( S_v^{\text{ord}}(\mathcal{N}; \mathcal{O}) \) of all \( T_0(n) \). Then,
there exist a \( \Lambda \)-algebra homomorphism \( \lambda : \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \to \mathfrak{P} \) and a point \( P \in \text{Spec}(\mathcal{F}(\mathfrak{o})) \) such that \( f \) is a constant multiple of \( f_P \). If \( f \) is a complex cusp form of weight \( k \geq 2t \), then \( P \) as above belongs to \( \mathcal{F}_{\text{alg}}(\mathcal{F}) \).

**Proof.** By definition, \( \lambda_P \) has values in the \( p \)-adic integer ring \( \mathfrak{o}' \) of a finite extension \( K'/K \). Since \( \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}') = \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \otimes_{\mathfrak{o}} \mathfrak{o}' \) by Theorems 3.1 and 3.3, we can extend \( \lambda \) to \( \lambda' : \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}') \to \mathfrak{P} \) by the combination:

\[
\mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \otimes_{\mathfrak{o}} \mathfrak{o}' \xrightarrow{\lambda \otimes \text{id}} \mathfrak{P} \otimes_{\mathfrak{o}} \mathfrak{o}' \xrightarrow{\psi} \mathfrak{P}
\]

Let \( \mathcal{X}' \) be the subfield of \( \mathfrak{P} \) generated by \( \mathfrak{o}' \) and \( \mathcal{X} \), and let \( \mathcal{F}' \) denote the integral closure of \( \Lambda \) in \( \mathcal{X}' \). Replacing \( \lambda \) and \( \mathcal{F} \) by \( \lambda' \) and \( \mathcal{F}' \), we may suppose that \( \lambda_P \) has values in \( \mathfrak{o}' \). Then by Theorem 5.3, we can find \( f_P \in S_v^{\text{ord}}(N; \mathfrak{o}) \) such that \( \lambda_P(h) = \langle h, f_P \rangle \) for all \( h \in \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \). Then we see that \( a(n, f_P) = \langle T_0(n), f_P \rangle = \lambda_P(T_0(n)) \) for each ideal \( n \), and

\[
a(m, f_P|T_0(n)) = \langle T_0(m), f_P|T_0(n) \rangle = \langle T_0(m)T_0(n), f_P \rangle = \lambda_P(T_0(m)T_0(n)) = \lambda_P(T_0(n))a(m, f_P)
\]

for ideals \( m \) and \( n \). This shows that \( f_P|T_0(n) = \lambda_P(T_0(n))f_P \). If \( P \in \mathcal{F}_{\text{alg}} \) and \( n(P) \geq 2v \), then \( \lambda_P \) factors through \( \mathcal{H}_{k, w}(Np^{\alpha(P)}; \mathfrak{e}_P; K) \) for \( k = n(P) - 2v + 2t \) and \( w = n(P) - v + t \) by Theorem 3.4 and hence factors through \( \mathcal{H}_{k, w}(Np^{\alpha(P)}; K) \). Since

\[
\mathcal{H}_{k, w}(Np^{\alpha(P)}; K) = \mathcal{H}_{k, w}(Np^{\alpha(P)}; K_0) \otimes_{K_0} K
\]

for a suitable finite extension \( K_0/\Phi(\mathfrak{e}) \) inside \( K \) and \( \mathcal{H}_{k, w}(Np^{\alpha(P)}; K_0) \) is of finite dimension over \( \mathcal{Q} \) by Theorem 3.1, the restriction of \( \lambda_P \) to \( \mathcal{H}_{k, w}(Np^{\alpha(P)}; K_0) \) has values in \( \mathcal{Q} \); in particular, \( \lambda_P(T_0(n)) \in \mathcal{Q} \) for all \( n \). Then by Corollary 5.2, \( f_P \) is contained in \( S_K^{\text{cusp}}(Np^{\alpha(P)}, \mathfrak{e}_P; \mathcal{C}) \).

Now we shall show the converse: If \( f \in S_v^{\text{ord}}(N; \mathfrak{o}) \) is a common eigenform of all Hecke operators \( T_0(n) \), then we can define an \( \mathfrak{o} \)-algebra homomorphism \( \mu : \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \to \mathfrak{o} \) by \( f \mid h = \mu(h)f \). Then, \( \text{Ker}(\mu) \) contains at least one minimal prime ideal \( \mathfrak{f} \) of \( \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \); namely, there exist a \( \Lambda \)-algebra homomorphism \( \lambda : \mathfrak{h}_v^{\text{ord}}(N; \mathfrak{o}) \to \mathfrak{P} \) with kernel \( \mathfrak{f} \) and a point \( P \in \text{Spec}(\mathcal{F}(\mathfrak{o})) \) such that \( \mu = P \circ \lambda = \lambda_P \). Then

\[
a(n, f) = \langle T_0(n), f \rangle + \langle T_0(n), f \rangle = \lambda_P(T_0(n))a(n, f) = a(n, f_P)a(n, f)
\]

for all \( n \).

This shows that \( f = a(n, f)f_P \), which finishes the proof.
6. A theorem of Matsushima and Shimura

In this section, we shall give an exposition of a result of Matsushima and Shimura [22] which relates the space of cusp forms to the cohomology groups of certain sheaves on modular varieties. In this and the following Sections 7, 8, 9 and 10, we treat general quaternion algebras \(B/F\) which may ramify at some places of \(F\). Fix a maximal order \(R\) of \(B\) as in Section 1. For each place \(\sigma\) of \(F\) outside \(\Sigma^B\), we identify \(R_\sigma = R \otimes_\mathbb{Q} \mathbb{Q}_\sigma\) with \(M_2(\mathbb{Q}_\sigma)\) if \(\sigma\) is finite and with \(M_2(\mathbb{R})\) if \(\sigma\) is infinite. Let \(U\) be an open compact subgroup of \(\tilde{R}^\times\). We define a subgroup \(G_p\) of \(G_p^B\) by \(G_p = \{x \in G^B_p | x \in U \} \) and \(U^p = \{ x \in U | x_p = 1 \}\). We shall decompose

\[
G^B_A = \bigcap_{i=1}^h G^B_{Q_i} t_i U C^B_{\infty} = \bigcap_{i=1}^h G^B_{Q_i} t_i U C^B_{\infty} \quad (G^B_Q = G^B_{Q_i} \cap G^B_{Q_i} G^B_{\infty})
\]

for \(t_i \in G^B_{A_i}\) such that \(t_i, \in \in C^B_{\infty}\) and \(t_i, p \in U^p\). As in Section 2, we put

\[
\Gamma_i(U) = t_i U C^B_{\infty} + t_i^{-1} \cap G^B_Q = t_i U C^B_{\infty} \cap G^B_Q,
\]

\[
\tilde{\Gamma}_i(U) = \Gamma_i(U) / F^\times \cap \Gamma_i(U)
\]

which are discrete subgroups of \(G^B_{\infty}\) and \(G^B_{\infty} / F^\times\). We define a complex analytic space \(X_i(U) = \Gamma_i(U) \setminus \mathcal{Z}_B^p (\mathcal{Z}_B = H^1_{\text{Galois}})\), which is a manifold if \(\tilde{\Gamma}_i(U)\) is without torsion and is compact if \(B\) is a division algebra. We shall construct sheaves on the modular variety \(X(U) = \mathcal{X}_Q \setminus G^B / U C^B_{\infty} = \mathcal{X}_Q \setminus G^B / U C^B_{\infty} + (G^B_A = G^B_{\infty} G^B_{\infty})\) out of a right module \(M\) of \(U_p\) or \(G^B_{\infty}\). The case where we consider right \(G^B_{\infty}\)-modules \(M\) (resp. right \(U_p\)-modules \(M\)) will be referred to as Case \(\infty\) (resp. Case \(p\)). We suppose that

(6.1) \(M\) is a finite dimensional real vector spaces in Case \(\infty\), and \(M\) or its Pontryagin dual module is a \(\mathbb{Z}_p\)-module of finite type in Case \(p\).

We let \(G^B_Q\) act on \(G^B_A \times M\) from the left by \(\alpha \cdot (g, m) = (\alpha g, m)\) and let \(U C^B_{\infty}\) act on it from the right by

\[
(g, m) \cdot u = \begin{cases} (gu, mu) & \text{in Case } \infty, \\ (gu, mu) & \text{in Case } p. \end{cases}
\]

Giving \(M\) the discrete topology, we consider the covering space of \(X(U)\):

\[
\mathcal{M}_\infty(U) = G^B_Q \setminus G^B_A \times M / U C^B_{\infty} + .
\]

We denote by \(\mathcal{M}\) the left \(U_p\) or \(G^B_{\infty}\) module whose underlying module is \(M\) but whose left action is given by \(u \cdot m = m \cdot u^{-1}\). Let \(\Gamma_i(U)\) act on \(\mathcal{M}\) through the natural inclusion: \(\Gamma_i(U) \hookrightarrow U_p\) in Case \(p\) and \(\Gamma_i(U) \hookrightarrow G^B_{\infty}\) in Case \(\infty\). Giving \(\mathcal{M}\) the discrete topology, we consider the covering space \(\mathcal{M}_\infty(U) = \)
\[ \Gamma'(U) \backslash X_B \times 'M, \] where we have let \( \Gamma'(U) \) act on \( X_B \times 'm \) from the left by the diagonal action.

**Proposition 6.1.** Put \( z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in X_B \). Then we have isomorphisms

(i) \( X(U) \cong \bigsqcup_{j-1}^h X_j(U) \) induced by \( G Q t_j U G_{\infty +} \ni \alpha t_j u \mapsto t_{j, \infty} u_{\infty}(z_0) \in X_B \),

(ii) \( M(U) \cong \bigsqcup_{j=1}^h M_j(U) \) induced by

\[ G Q t_j U G_{\infty +} \times M \ni (m, \alpha) \mapsto (t_{j, \infty} u_{\infty}(z_0), t_{j, \sigma} u_{\sigma} m) \in X_B \times 'M, \]

where \( \sigma = p \) or \( \infty \) according as we are in Case \( p \) or in Case \( \infty \).

**Proof.** The assertion (i) is well-known. We shall prove (ii) only in Case \( p \) because Case \( \infty \) can be treated similarly. Define \( \phi : G Q t_j U G_{\infty +} \times M \to X_B \times 'M \) as in the second assertion. Then, for \( \beta \in G Q \) and \( s \in U G_{\infty +} \), we know that

\[ \phi(\beta \alpha t_j u s, m_0 \cdot s_p) = (t_{j, \infty} u_{\infty}(z_0), t_{j, \sigma} u_{\sigma} s_p (s_p^{-1} m_0)) = (t_{j, \infty} u_{\infty}(z_0), t_{j, \sigma} u_{\sigma} m). \]

If \( \alpha t_j u = t_j u' \) for \( u, u' \in U G_{\infty +} \) and \( \alpha \in G Q \), then \( \alpha = t_j u' u^{-1} t_j^{-1} \) and thus \( \alpha \in \Gamma'(U) \). Writing \( m = t_{j, p} u_p m_0 \) and \( z = t_{j, \infty} u_{\infty}(z_0) \), we have

\[ \phi(t_j u', m_0) = (t_{j, \infty} u'_{\infty}(z_0), t_{j, \sigma} u'_{\sigma} m_0) = (\alpha(z), \alpha m) \]

since \( \alpha = t_j u'(t_j u)^{-1} \).

This shows that \( \phi \) induces the isomorphism in (ii).

If \( \Gamma'(U) \) is without torsion and if \( \Gamma'(U) \cap F^\times \) acts trivially on \( 'M \), \( M_j(U) \) is locally isomorphic to the manifold \( X_j(U) \), and we can consider the sheaf of continuous sections of \( M_j(U) \) on \( X_j(U) \). This sheaf will be denoted by the same symbol \( M_j(U) \). Let \( K/\mathbb{Q}_p \) be a finite extension in \( \overline{\mathbb{Q}}_p \) containing \( K_0 \) as in (1.1). Let \( \mathcal{O} \) denote the \( p \)-adic integer ring of \( K \). Then, we consider the sheaves on \( X(U) \) or \( X_j(U) \) corresponding to the module \( L(n, v; A) \) for any \( \mathcal{O} \)-algebra \( A \) in Case \( p \) and any \( \mathbb{R} \)-algebra \( A \) containing \( K_0 \) in Case \( \infty \). The corresponding sheaf will be denoted by

\[ \mathcal{L}(n, v; A)_{/X(U)} \] and \( \mathcal{L}(n, v; A)_{/X_j(U)} \)

(under the condition: \( n + 2v \sim 0 \), \( \Gamma'(U) \cap F^\times \) acts trivially on \( L(n, v; A) \) either if \( U \subset U_j(p) \) for odd prime \( p \) or if \( U \subset U_j(4) \)). For simplicity, we shall assume that \( t_{1, \infty} = 1 \) for all \( i \), and for \( f \in S_{k, w, j}(U; C) (J \subset I_B) \), we write \( f_i \) for \( f_{t_i} \) as in (2.4b). Then, we know that \( f_i \in S_{k, w, j}(\Gamma'(U); C) \). For each subset \( J \) of \( I_B \), we put \( \bar{J} = \{ \tau \in I_B | \tau \notin J \} \) and \( dz_j = (\Lambda_{\tau \in J} dz_\tau) \wedge (\Lambda_{\tau \in \bar{J}} d\bar{z}_\tau) \) as a differential on \( X_B \). For \( \gamma \in G_{\infty +} \),

\[ (6.2a) \quad dz_j \circ \gamma = f_j(\gamma, z)^{-2t_B p(\gamma) t_n} dz_j. \]
We define for $n = \Sigma \tau \cdot n_\tau \in \mathbb{Z}[I],$
\[
\eta_n(z) = \prod_{\tau \in I} \eta_n(z_\tau) \prod_{\tau \in f} \eta_n(z_\tau) \in \mathbb{I}L(n_B, v_B; C) \quad \text{for } z \in \mathcal{D}_B \text{ by }
\]
\[
\eta_n(z_\tau) = \sum_{i=0}^{n_\tau} (-z_\tau)^i X^{n_\tau-i} Y^i = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)^n \tau \left(\begin{array}{c} X_\tau \\ Y_\tau \end{array}\right),
\]
where we put $\left(\begin{array}{c} X \\ Y \end{array}\right)^m = (X^m, X^{m-1}Y, \ldots, Y^m)$ for $0 \leq m \in \mathbb{Z}.$ Then we have
\[
(6.2b) \quad \gamma \cdot \eta_n(z) = \eta_n(\gamma(z)) \nu(\gamma)^{-n_\tau} j(\gamma, z)^{n_\tau} \quad \text{for all } \gamma \in \Gamma_\infty.
\]
For each $f \in S_{k, w, f}(\Gamma_i(U); B; C),$ we put
\[
\omega(f) = f(z) \cdot \eta_n(z) \, dz_f,
\]
which is a harmonic $r$-differential form on $\mathcal{D}_B$ with values in $\mathbb{I}L(n, v; C)$ in the sense of [22]. Since $n = k - 2t$ and $v = w - k + t,$ we see easily from (6.2a, b) the $\Gamma_i(U)$-invariance of $\omega(f):$
\[
(6.3) \quad \omega(f) \circ \gamma = \gamma \cdot \omega(f) \quad \text{for } \gamma \in \Gamma_i(U).
\]
Thus if $\tilde{\Gamma}_i(U)$ is without torsion, we can regard $\omega(f)$ as an $r$-differential form with values in the sheaf $\mathcal{L}_i(n, v; C)/X_i(U).$ Thus by assigning the de Rham cohomology class of $\omega(f)$ to $f \in S_{k, w, f}(\Gamma_i(U); C),$ we have a morphism:
\[
S_{k, w, f}(\Gamma_i(U); C) \to H^r(X_i(U), \mathcal{L}_i(n, v; C)).
\]
When $\tilde{\Gamma}_i(U)$ has non-trivial torsion elements, by choosing a sufficiently small normal subgroup $\Gamma$ of $\tilde{\Gamma}_i(U)$ of finite index without torsion, with a slight abuse of symbols, we write, for a field $K$ of characteristic 0,
\[
H^r(X_i(U), \mathcal{L}_i(n, v; K)) \quad \text{for } H^r(\Gamma \setminus \mathcal{D}_B, \mathcal{L}(n, v; K))^{\Gamma_i(U)},
\]
where $\mathcal{L}(n, v; K)/(\Gamma \setminus \mathcal{D}_B)$ is the sheaf on $\Gamma \setminus \mathcal{D}_B$ defined in exactly the same manner as $\mathcal{L}(n, v; K)/X_i(U).$ This space is determined independently of the choice of $\Gamma.$ Then, the above map induces a morphism: $S_{k, w, f}(U; C) \to H^r(X(U), \mathcal{L}(n, v; C)).$ If $r = 2s$ ($r = |I_B| = \dim_{\mathbb{C}} \mathcal{D}_B$ with $0 < s \in \mathbb{Z},$ we put, for each $J \subset I_B$ with $|J| = s,$
\[
\omega_j = \bigwedge_{\tau \in J} \text{Im}(z_\tau) \cdot dz_\tau \wedge d\bar{z}_\tau,
\]
\[
\text{Inv}(X_j(U)) = \sum_{|J| = s} C \omega_{I/X_j(U)} \in H^r(X_j(U), C),
\]
\[
\text{Inv}(U) = \bigoplus_{j} \text{Inv}(X_j(U)).
\]
When $r = 0$ (i.e., $B$ is totally definite), we denote by $\text{Inv}(U)$ the space defined in (2.4d) under the same symbol. Then we have the following result of Matsushima and Shimura:

**Theorem 6.2 ([22, §4]).** Suppose that $n + 2v \sim 0$ and $n \geq 0$ and that $B$ is a division algebra. Put $k = n + 2t$ and $w = v + k - t$. Then, we have a canonical isomorphism induced by $f \mapsto \omega(f)$:

$$H^r(X(U), \mathcal{L}(n, v; C)) \cong \begin{cases} \text{Inv}(U) \oplus \left( \bigoplus_{j \in I_B} S_{k, w, j}(U; C) \right) & \text{if } r \text{ is even and } n = 0, \\ \bigoplus_{j \in I_B} S_{k, w, j}(U; C) & \text{if either } r \text{ is odd or } n > 0. \end{cases}$$

In fact, in [22, §4], the case where $B$ is indefinite is studied, but the totally definite case follows from (2.6b). This result has already been generalized by Harder ([5], [6]; see also [7]) even for $B = M_2(F)$ with an appropriate modification, but we will not need this general fact later.

We shall now give a definition of sheaves $\mathcal{L}(n, v; A)$ for global $z(v)$-algebras $A$. Let $L$ be a finite extension of the field $K_0$ as in Section 1, and let $A$ be the integer ring of $L$. Since $G^B_f$ naturally acts on $L(n, v; L_f)$ and hence on $^tL(n, v; L_f)$ (here $L_f$ is the finite part of the adele ring $L_A$ of $L$). We can consider $t_i \cdot ^tL(n, v; A)$ and

$$^tL_i(n, v; A) = t_i \cdot ^tL(n, v; A) \cap ^tL(n, v; L),$$

where $\hat{A}$ denotes the compact ring $A \otimes_{Z} \hat{Z}$. Then naturally, $\Gamma^i(U) = t_iUG_{\infty} + t_i^{-1} \cap G_Q$ acts on $^tL_i(n, v; A)$. If $\bar{\Gamma}(U)$ is without torsion, then we can consider the sheaf

$$\mathcal{L}_i(n, v; A) = \bar{\Gamma}(U) \setminus Z_B \times ^tL_i(n, v; A) \text{ over } X_i(U),$$

$$\mathcal{L}(n, v; A) = \bigsqcup_{i=1}^h \mathcal{L}_i(n, v; A) \text{ over } X(U).$$

There is a canonical and functorial isomorphism

$$(6.4) \quad H^r(X(U), \mathcal{L}(n, v; A)) \cong \bigoplus_{i=1}^h H^r(\bar{\Gamma}(U), ^tL_i(n, v; A)), $$

where the right-hand side is the group cohomology group for the $\bar{\Gamma}(U)$-module $^tL_i(n, v; A)$. Note that the right-hand side of (6.4) is defined without the assumption of torsion-freeness of $\bar{\Gamma}(U)$.
For each \( A \)-algebra \( D \), we define the sheaf \( \mathcal{L}(n, v; D) \) on \( X(U) \) by
\[
\mathcal{L}(n, v; D)_{/X(U)} = \mathcal{L}(n, v; A) \otimes_A D_{/X(U)}.
\]

**Theorem 6.3.** Suppose that \( \Gamma^i(U) \) is without torsion for all \( i = 1, \ldots, h \). Let \( D \) be an \( A \)-algebra. Suppose one of the following conditions holds:
(i) \( D \) is the integer ring of a finite extension of \( L \);
(ii) \( D \) is a field extension of \( L \);
(iii) \( D \) is a localization of \( A \).
Then, we have a canonical isomorphism:
\[
H^r_c(X(U), \mathcal{L}(n, v; D)) \cong H^r_c(X(U), \mathcal{L}(n, v; A)) \otimes_A D,
\]
where \( H^r_c \) means the cohomology group with compact support.

By this theorem, the natural image of \( H^r_c(X(U), \mathcal{L}(n, v; A)) \) gives an \( A \)-integral structure on \( H^r_c(X(U), \mathcal{L}(n, v; \mathbb{C})) \) and on \( H^r_c(X(U), \mathcal{L}(n, v; \mathbb{Q}_p)) \).

**Proof.** We firstly suppose that \( D \) is the localization of \( A \) by a multiplicative set \( S \subset A \). For each \( 0 \neq a \in S \), we consider an \( A \)-module \( a^{-1}A \subset L \). Then we have an exact sequence of \( A \)-modules:
\[
0 \to L_i(n, v; A) \to L_i(n, v; a^{-1}A) \to L_i(n, v; a^{-1}A/A) \to 0.
\]
This gives another exact sequence:
\[
H^{r-1}_c(X(U), \mathcal{L}(n, v; a^{-1}A/A)) \rightarrow H^r_c(X(U), \mathcal{L}(n, v; A))
\rightarrow H^r_c(X(U), \mathcal{L}(n, v; a^{-1}A))
\rightarrow H^r_c(X(U), \mathcal{L}(n, v; a^{-1}A/A))
\rightarrow \cdots.
\]
By taking the injective limit relative to \( a \in S \), we have another exact sequence:
\[
\lim_{\rightarrow a} H^{r-1}_c(X(U), \mathcal{L}(n, v; a^{-1}A/A))
\rightarrow H^r_c(X(U), \mathcal{L}(n, v; A))
\rightarrow H^r_c(X(U), \mathcal{L}(n, v; D))
\rightarrow \lim_{\rightarrow a} H^r_c(X(U), \mathcal{L}(n, v; a^{-1}A/A)).
\]
Any element in the modules at the extreme right and left of the above sequence is killed by some element in \( S \). Therefore, their tensor product with \( D \) becomes trivial. Thus by tensoring \( D \) to the above sequence, we have
\[
H^r_c(X(U), \mathcal{L}(n, v; A)) \otimes_A D \cong H^r_c(X(U), \mathcal{L}(n, v; D)) \otimes_A D
\equiv H^r_c(X(U), \mathcal{L}(n, v; D)).
\]
since $D$ is $A$-flat. Secondly, let $\mathfrak{p}$ be a prime ideal of $A$, and let $A_{\mathfrak{p}}$ be the localization of $A$ at $\mathfrak{p}$. As we have already seen,

$$H'_c(X(U), \mathcal{L}(n, v; A_{\mathfrak{p}})) \cong H'_c(X(U), \mathcal{L}(n, v; A)) \otimes_A A_{\mathfrak{p}}.$$  

On the other hand, by the universal coefficient theorem in [2, II, Th. 18.3], for any flat $A_{\mathfrak{p}}$-algebra $D$, we have

$$H'_c(X(U), \mathcal{L}(n, v; A)) \otimes_A D \cong H'_c(X(U), \mathcal{L}(n, v; A_{\mathfrak{p}})) \otimes_A D \cong H'_c(X(U), \mathcal{L}(n, v; D)).$$

This result includes the Case (ii). Finally, assuming the first condition (i), we have a natural map: $H'_c(X(U), \mathcal{L}(n, v; A)) \otimes_A D \rightarrow H'_c(X(U), \mathcal{L}(n, v; D))$. After localizing this map at each prime ideal $\mathfrak{p}$, we get an isomorphism by the results already proved. Therefore the original map must be an isomorphism since these cohomology groups are of finite type as modules over $A$ or $D$ (see e.g. [28]).

The type of sheaves over $X(U)$ discussed here was first considered by Langlands (see e.g. [21]), but the definition of these sheaves using local action of $U_p$ or $G_{\infty+}$ is newly adopted here under some influence of Harder [6, p. 131] and the work of Matsushima and Murakami which precedes [22] and is very well suited to $p$-adic arguments. We may also note that the definition in Case $p$ reminds us of a work of Weil [37] in the case of GL(1).

7. Hecke operators on cohomology groups and proof of Theorems 3.1 and 4.10

We shall firstly define Hecke operators on $H^i(X(U), \mathcal{L}(n, v; A))$ and on $\bigoplus_{i=1} H^i(\Gamma(U), L_i(n, v; A))$. After that, we shall give a proof of Theorems 3.1 and 4.10. Let $\sigma$ be either the place $p$ or $\infty$ of $\mathbb{Q}$. Let $\Delta$ be a multiplicative semi-group inside $R_p = R \otimes \mathbb{Z}$. Let $U$ and $U'$ be two open compact subgroups of $G_f^B$ such that $U_p \subseteq \Delta$ and $U'_p \subseteq \Delta$ if $\sigma = p$. Suppose that $\Gamma(U)$ and $\Gamma(U')$ are torsion-free for all $i$ when we consider sheaf cohomology groups. Let $M$ be a right $\Delta$-module or a right $G_{\infty+}$-module according as $\sigma = p$ or $\infty$ satisfying (6.1). We suppose that $E$ acts trivially on $M$ in order to have the non-trivial sheaf $\mathcal{M}(U)$. Let $x \in G_f C_{\infty}$ such that $x_p \in \Delta$ if $\sigma = p$. Write $U^x = x^{-1} U x_f$ in $G_f^B$, and put $V = U \cap (U')^{-1}$. Then $V^x = U^x \cap U'$. Now we shall define a morphism

$$[x] : G_A \times M \rightarrow G_A \times M \quad \text{by} \quad [x](g, m) = (gx, m \cdot x_\sigma).$$

Then we see that, for $a \in G_Q$ and $u \in VC_{\infty+}$,

$$[x](ag_u, mu_\sigma) = (agux, mu_\sigma x_\sigma) = (agux^x, m_\sigma u^x_\sigma),$$
where \( u^x = x^{-1}ux \). Thus the map \([x]\) induces a morphism of sheaves \([x]\): \( \mathcal{M}(V) \to \mathcal{M}(V^x) \), and we then have the induced morphism:

\[
(7.1a) \quad [x]: H^q(X(V), \mathcal{M}(V)) \to H^q(X(V^x), \mathcal{M}(V^x)).
\]

Let \( \text{pr}: X(V^x) \to X(U') \) be the natural projection, and consider the sheaf \( \mathcal{F} = (\text{pr})_* (\mathcal{M}(U')) = (\text{pr})_* (\mathcal{M}(V^x)) \). We take a Galois (étale) covering \( \pi: Y \to X(U') \) which factors \( X(V^x) \). Put

\[
\mathcal{F}' = \pi_* \pi^* (\mathcal{M}(U')),
\]

\( \mathcal{G} = \text{Gal}(Y/X(U')) \)

and

\[
\mathcal{H} = \text{Gal}(Y/X(V^x)).
\]

Since \( \text{pr} \) is an open map (in fact, it is a local isomorphism), for each open set \( O \subset X(U') \), we have that \( \Gamma(O, \mathcal{F}) = \Gamma(\text{pr}^{-1}(O), \mathcal{M}(V^x)) \). If \( O \) is sufficiently small, then there are disjoint open subsets \( \{ O_\tau \}_{\tau \in \mathcal{G}} \) of \( Y \) such that \( \pi \) induces \( O_\tau \cong O \) for each \( \tau \). Then

\[
\mathcal{F}'(O) = \bigoplus_{\tau \in \mathcal{G}} \pi^* (\mathcal{M}(U'))(O_\tau) \cong \bigoplus_{\tau \in \mathcal{G}} \Gamma(O, \mathcal{M}(U'))
\]

and

\[
H^0(\mathcal{G}, \mathcal{F}'(O)) = \Gamma(O, \mathcal{M}(U')).
\]

On the other hand, we have a natural morphism: \( \mathcal{F} \to \mathcal{F}' \). For \( s \in \mathcal{F}(O) \), we define \( \text{Tr}(s) = \sum_{\tau \in \mathcal{G}} \sigma(s) \in H^0(\mathcal{G}, \mathcal{F}'(O)) \). This extends to a morphism of sheaves \( \text{Tr}: \mathcal{F} \to \mathcal{M}(U') \). By definition, \( H^q(X(U'), \mathcal{F}') = H^q(X(V^x), \mathcal{M}(V^x)) \), and we thus obtain the trace map

\[
(7.1b) \quad \text{Tr}^{-v}_{U'/V^x}: H^q(X(V^x), \mathcal{M}(V^x)) \to H^q(X(U'), \mathcal{M}(U')).
\]

We also have the restriction morphism:

\[
(7.1c) \quad \text{res}^{-v}_{U'/V}: H^q(X(U), \mathcal{M}(U)) \to H^q(X(V), \mathcal{M}(V)).
\]

We shall define \([UxU']\): \( H^q(X(U), \mathcal{M}(U)) \to H^q(X(U'), \mathcal{M}(U')) \) by

\[
(7.2a) \quad [UxU'] = \text{Tr}^{-v}_{U'/V^x} \circ [x] \circ \text{res}^{-v}_{U'/V}.
\]

Now we shall extend a little the definition of the module \( L(n, v; A) \). Let \( K/Q_p \) be a finite extension containing \( K_0 \) as in Section 1 and \( \mathcal{O} \) be its \( p \)-adic integer ring. Suppose that \( N \) is prime to \( p \) and that every prime factor of \( Np \) is unramified in \( B \). Put for \( \beta \geq \alpha \geq 0 \),

\[
\Delta^\alpha_\beta(N) = \left\{ x \in \hat{R} \left| x_{Np} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \text{ with } a-1 \in p^\alpha N^\beta_{Np}, \ a \in \hat{R}^\times_{Np}, \ \text{and } c \in p^\beta N^\alpha_{Np}, \ \text{and } x \in G_f \right. \right\},
\]

\[
U^\alpha_\beta(N) = \hat{R}^\times \cap \Delta^\alpha_\beta(N), \quad V^\alpha_\beta(N) = \hat{R}^\times \cap \Delta^\alpha_\beta(N)^1, \quad \Delta^\alpha_\beta = \Delta^\alpha_\beta(1),
\]
where \( \Delta^\alpha_\beta(N)^i = \{ x^i | x \in \Delta^\alpha_\beta(N) \} \). Let \( A \) be an \( \mathcal{O} \)-algebra and \( \lambda: \mathbb{Z}_a/\mathbb{Z}_\beta \to A^\times \) be a character such that \( \lambda x_{n+2v} \) induces a character of \( \mathbb{Z}_a \to A \). (This is trivially true if \( \alpha \) is sufficiently large.) We shall now twist the action of \( \Delta^\alpha_\beta \) on \( L(n, v; A) \) by the character \( \lambda \): We let \( u \in \Delta^\alpha_\beta \) with \( u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) act on \( m \in L(n, v; A) \) by

\[
m \cdot u = \lambda(a)(m \cdot u_p),
\]

where the action of \( u_p \) in the parentheses of the right-hand side is the usual action of \( u_p \) on \( L(n, v; A) \). Similarly, we can let \( u \in (\Delta^\alpha_\beta)^i \) with \( u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) act on \( L(n, v; A) \) by \( m \cdot u = \lambda(d)(m \cdot u_p) \). The \( \Delta^\alpha_\beta \)-module (resp. \( (\Delta^\alpha_\beta)^i \)-module) \( L(n, v; A) \) with this twisted action will be denoted by \( L(n, v, \lambda; A) \) (resp. \( L^*(n, v, \lambda; A) \)). Thus we can define the action of \([U^\alpha_\beta(N)xU^\alpha_\beta(N)]\) for \( x \in \Delta^\alpha_\beta \) or \([V^\alpha_\beta(N)xV^\alpha_\beta(N)]\) for \( x \in (\Delta^\alpha_\beta)^i \) on the corresponding cohomology groups. The corresponding sheaves to \( L(n, v, \lambda; A) \) and \( L^*(n, v, \lambda; A) \) on \( X(U) \) \((U \subset U^\alpha_\beta(N) \text{ or } U \subset V^\alpha_\beta(N))\) will be denoted by \( \mathcal{L}(n, v, \lambda; A) \) and \( \mathcal{L}^*(n, v, \lambda; A) \).

Let \( U \) and \( U' \) be two open compact subgroups of \( \Delta^\alpha_\beta \). We shall now modify the operators \([UxU'] \) \((x \in \Delta^\alpha_\beta)\) analogously to the definition of \( T_\theta(n) \) out of \( T(n) \). To define the morphism \([x] \) for \( x \in \Delta^\alpha_\beta \), we have used the action of \( x_p \) on \( L(n, v, \lambda; A) \), which will be written as \( m \mapsto m \cdot x_p \). Since the underlying \( \mathcal{A} \)-modules of \( L(n, v, \lambda; A) \) and \( L(n, 0, \lambda; A) \) are the same, we may use the action of \( x_p \) on \( L(n, 0, \lambda; A) \), which will be written as \( m \mapsto m \circ x_p \), to define a map similar to \([x] \):

\[
(x): H^q(X(V), \mathcal{L}(n, v, \lambda; A)) \to H^q(X(V^x), \mathcal{L}(n, v, \lambda; A)).
\]

Note that for \( x \in \Delta^\alpha_\beta \), \( m \cdot x_p = \det(x_p)^m \circ x_p \). The \( \mathcal{O} \)-algebra \( A \) always satisfies the condition (3.1) since \( \mathcal{O} \) does. We fix a character: \( F^\alpha_x \ni a \mapsto \{ a^v \} \in A \) as in Section 3. Then \( \{ v(x)^{-v} \} \det(x_p)^v \) is a unit in \( A \), and thus we can define a morphism of sheaves

\[
(x): \mathcal{L}(n, v, \lambda; A)_{/X(V)} \to \mathcal{L}(n, v, \lambda; A)_{/X(V^x)},
\]

by the correspondence: \((g, m) \mapsto (gx, \{ v(x)^{-v} \} \det(x_p)^v(m \circ x_p)) \) for \( g \in G^B_\lambda \) and \( m \in L(n, v, \lambda; A) \). This induces a linear map

\[
(x): H^q(X(V), \mathcal{L}(n, v, \lambda; A)) \to H^q(X(V), \mathcal{L}(n, v, \lambda; A)).
\]

We then define \((UxU') : H^q(X(U), \mathcal{L}(n, v, \lambda; A)) \to H^q(X(U'), \mathcal{L}(n, v, \lambda; A)) \) by

\[
(7.2b) \quad (UxU') = \Tr_{U'/V'} \circ (x) \circ \text{res}_{U/V}.
\]

One can formulate the above definitions for \( L^*(n, v, \lambda; A) \) and \( L^*(n, v, \lambda; A) \) for \( x \in (\Delta^\alpha_\beta)^i \) and \( U, U' \subset (\Delta^\alpha_\beta)^i \) in exactly the same manner as above:

\[
(UxU') : H^q(X(U), \mathcal{L}^*(n, v, \lambda; A)) \to H^q(X(U'), \mathcal{L}^*(n, v, \lambda; A)).
\]
Then we have the relation
\[(7.3) \quad [UxU'] = \{v(x)^v\}(UxU') \quad \text{for all } x \in \Delta_\beta \quad \text{or} \quad x \in (\Delta_\beta')'.\]

The operators $[UxU']$ and $(UxU')$ depend only on the double coset $UC_\infty+xU'C_\infty+$ (and the choice of the map: $a \mapsto \{a^v\}$) and are independent of the choice of $x$ in $UC_\infty+xU'C_\infty+$. Furthermore, one can verify
\[
[U'xU''] \circ [UyU'] = [UyU' \cdot U'xU''], \quad (U'xU'') \circ (UyU') = (UyU' \cdot U'xU''),
\]
where the product on the right-hand side is taken in the abstract Hecke ring as in [36, III]. By decomposing
\[
\{ x \in \Delta_\beta'(N) \mid v(x) = n \} = \bigsqcup_i U_\beta'(N)x_iU_\beta'(N),
\]
we can define Hecke operators $T(n)$ and $T_\partial(n)$ by
\[
T(n) = \sum_i [U_\beta'(N)x_iU_\beta'(N)], \quad T_\partial(n) = \sum_i (U_\beta'(N)x_iU_\beta'(N))
\]
on $H^q\left(X(U_\beta'(N)), \mathcal{L}(n, v, \lambda; A)\right)$.

So far, we have only considered sheaves defined locally at $\sigma$. Now we shall extend our definitions of operators $(UxU')$ to global rings $A$. Let $L$ be a finite extension of $K_0$, and let $A$ be the integer ring of $L$. We suppose the condition (3.1) for $A$. Decompose $G_\Lambda = \bigsqcup t_jG_\Lambda t_j^{-1}VG_\infty+$ and $G_\Lambda^B = \bigsqcup t_jG_\Lambda t_j^{-1}'VG_\infty^B +$. Suppose that $t_{j, \infty} = t_{j, \infty}' = 1$. We put $L_j(n, v; A) = t_j \cdot L_j(n, v; \hat{A}) \cap L(n, v; L)$ and $L_j'(n, v; A) = t_j \cdot L_j'(n, v; \hat{A}) \cap L'(n, v; \hat{A})$, where the intersection is taken in $\hat{L}(n, v; L_f)$ and $\widehat{\Lambda} = A \otimes _Z \hat{\mathbb{Z}} \subset L_f$. Expressing $t_jx = \gamma_jt_j'u^v$ for $x \in (G_f \cap \hat{\Gamma} \cap C_\infty$ with $u \in VG_\infty+$ and $\gamma_j \in G_\Lambda$, we have that $\gamma_j = t_ju^{-1}t_j'x^{-1} \in t_j \hat{\Gamma}t_j'^{-1}$, and thus $\gamma_j$ induces a linear map:
\[
\begin{array}{ccc}
L_j(n, v; A) & \longrightarrow & L_j'(n, v; A) \\
m \mapsto & \gamma_j^{-1}m = m \cdot \gamma_j.
\end{array}
\]

Note that
\[
\Gamma'(V^\ast) = t_j'V^\ast G_\infty + t_j'^{-1} \cap G_\Lambda = \gamma_j^{-1}t_jV^\ast G_\infty + t_j'^{-1} \gamma_j \cap G_\Lambda = \gamma_j^{-1}\Gamma'(V)\gamma_j,
\]
Then we have morphisms of sheaves:
\[
[x]_j : \mathcal{L}_j(n, v; A)_{/X_j(V)} \to \mathcal{L}_j'(n, v; A)_{/X_j'(V)},
\]
induced by $(z, m) \mapsto \left(\gamma_j^{-1}(z), \gamma_j^{-1}m\right)$,
\[
(x)_j : \mathcal{L}_j(n, v; A)_{/X_j(V)} \to \mathcal{L}_j'(n, v; A)_{/X_j'(V)},
\]
induced by $(z, m) \mapsto \left(\gamma_j^{-1}(z), \{v(x)^v\}(\gamma_j^{-1}m)\right).
Here, actually, \( \{ v(x)^{-v} \} (\gamma_j^{-1} m) = (\{ v(x)^{-v} \} v(\gamma_j)^v (m \circ \gamma_j)) \) is a well-defined element of \( 'L'_j(n, v; A) \) since \( (\{ v(x)^{-v} \} v(\gamma_j)^v) \) is a unit in \( A \). We also have morphisms of group cohomology:

\[
[x]_j: H^q(\Gamma_i(V), 'L_j(n, v; A)) \to H^q(\Gamma_i(V^x), 'L'_i(n, v; A))
\]

\[
(x)_j: H^q(\Gamma_i(V), 'L_j(n, v; A)) \to H^q(\Gamma_i(V^x), 'L'_i(n, v; A))
\]

which are given by

\[
\xi[x]_j(\alpha_0, \ldots, \alpha_q) = \gamma_j^{-1} \xi(\gamma_j \alpha_0 \gamma_j^{-1}, \ldots, \gamma_j \alpha_q \gamma_j^{-1}),
\]

\[
\xi(x)_j(\alpha_0, \ldots, \alpha_q) = (\{ v(x)^{-v} \} v(\gamma_j)^v) \xi(\gamma_j \alpha_0 \gamma_j^{-1}, \ldots, \gamma_j \alpha_q \gamma_j^{-1}) \circ \gamma_j
\]

for each \( q \)-homogeneous cocycle \( \xi: \Gamma_i(U) \to 'L_j(n, v; A) \).

In order to compare this definition of \([x]_j \) and \((x)_j \) with those of \([x] \) and \((x) \), we denote by \( A_\sigma \) the complex field \( \mathbb{C} \) or the closure of \( A \) in \( \overline{Q}_p \) according as \( \sigma = \infty \) or \( \sigma = p \). We can define the sheaf \( \mathcal{L}(n, v; A_\sigma) \) locally at \( \sigma \) as in the beginning of this section. By Proposition 6.1, we have natural isomorphisms:

\[
\phi: \mathbb{G}_Q \setminus \mathbb{G}_Q t_j V \mathbb{G}_{\infty +} \times L(n, v; A_\sigma) / V \mathbb{G}_{\infty +} = \Gamma_i(V) \setminus \mathbb{Z}_p^\times \times L(n, v; A_\sigma)
\]

\[
(\alpha t_j u, m_0) \quad \mapsto \quad (t_j \infty u(z_0), t_j \sigma u \sigma m_0),
\]

\[
\phi': \mathbb{G}_Q \setminus \mathbb{G}_Q t'_j V^x \mathbb{G}_{\infty +} \times L(n, v; A_\sigma) / V^x \mathbb{G}_{\infty +} = \Gamma_i(V^x) \setminus \mathbb{Z}_p^\times \times L(n, v; A_\sigma)
\]

\[
(\alpha t'_j u^x, m_0) \quad \mapsto \quad (t'_i \infty u^x(z_0), t'_i \sigma u' \sigma m_0).
\]

Since \( t_j x = \gamma_j t'_i u^x \), we have for any \( u' \in V \mathbb{G}_{\infty +} \), \( t_j u' x = \gamma_j t'_i (uu')^x \) and thus \( G_Q t_j V \mathbb{G}_{\infty +} x = G_Q t'_i V^x \mathbb{G}_{\infty +} \). This means that the map

\[
[x]: \mathcal{L}(n, v; A_\sigma) / X(V) \to \mathcal{L}(n, v; A_\sigma) / X(V^x)
\]

induces a morphism \([x]_j: \mathcal{L}(n, v; A_\sigma) / X_j(V) \to \mathcal{L}(n, v; A_\sigma) / X_j(V^x) \). We shall compute \( \phi' \circ [x] \circ \phi^{-1} \). To do this, write

\[
z = t_j \infty u'_i (z_0)
\]

and

\[
m = t_j \sigma u' \sigma m_0.
\]

Then

\[
\phi(g, m_0) = (z, m) \quad \text{for} \quad g = \alpha t_j u' \quad (\alpha \in G_Q \text{ and } u' \in V \mathbb{G}_{\infty +}).
\]
On the other hand, we see that
\[
\phi'(\{x\}, m_0) = \phi'(at_j u', m_0 x_\sigma)
\]
\[
= \phi'(\alpha \gamma_j t_i' (uu')^g, m_0 x_\sigma)
\]
\[
= \left( (t_i' (uu')^g)_\infty (z_0), (t_i' (uu')^g)_\sigma \cdot x_\sigma^{-1} m_0 \right) = (\gamma_j^{-1}(z), \gamma_j^{-1} m),
\]
since \( t_j u' = \gamma_j t_i' (uu')^g \). This shows
\[
(7.4a) \quad \phi' \circ [x] = [x]_j \circ \phi \quad \text{and} \quad \phi' \circ (x) = (x)_j \circ \phi.
\]
Similarly, when \( \tilde{\Gamma}'(U) \) and \( \tilde{\Gamma}'(U') \) are torsion-free, we have a commutative diagram:
\[
H^q(X_j(V), \mathcal{L}_j(n, v; A)) \cong H^q(\tilde{\Gamma}'(V), ^t L(n, v; A))
\]
(7.4b) \[
H^q(X_j(V^x), \mathcal{L}_j'(n, v; A)) \cong H^q(\tilde{\Gamma}'(V^x), ^t L_j'(n, v; A)),
\]
where the horizontal isomorphisms are the canonical ones (see e.g. [28, §2]).

Since the covering \( \pi_x: X(V^x) \to X(U) \) and \( \pi: X(V) \to X(U) \) are étale finite and \( \pi_x^*(\mathcal{L}(n, v; A)/X(U')) = \mathcal{L}(n, v; A)/X(V^x) \), we have
\[
\text{Tr}_{V'/V^x}: H^q(X(V^x), \mathcal{L}(n, v; A)) \to H^q(X(U'), \mathcal{L}(n, v; A)),
\]
\[
\text{res}_{U/V^x}: H^q(X(U), \mathcal{L}(n, v; A)) \to H^q(X(V), \mathcal{L}(n, v; A)).
\]
Define \([x]\) and \((x): \ H^q(X(V), \mathcal{L}(n, v; A)) \to H^q(X(V^x), \mathcal{L}(n, v; A))\) by \([x] = \bigoplus_j [x]_j\) and \((x) = \bigoplus_j (x)_j\), and put
\[
[U x U'] = \text{Tr}_{U'/V^x} \circ [x] \circ \text{res}_{U/V}, \quad (U x U') = \text{Tr}_{U'/V^x} \circ (x) \circ \text{res}_{U/V}.
\]
Then we see from (7.4) the compatibility between the previous definition and the new one.

Without assuming the torsion-freeness of \( \tilde{\Gamma}'(U) \) and \( \tilde{\Gamma}'(U') \), we can define
\[
[U x U'] \quad \text{and} \quad (U x U'): \bigoplus_i H^q(\tilde{\Gamma}'(U), ^i L_i(n, v; A))
\]
\[
\to \bigoplus_i H^q(\tilde{\Gamma}'(U'), ^i L_i'(n, v; A)).
\]
When \( B \) is totally definite, the non-trivial cohomology group is obtained only when \( q = 0 \), and the space of cohomology groups as above is nothing but the space of functions on \( G^B_A \) with values in \( L(n, v; A) \) on \( G^B_f \) satisfying (2.4a) by (2.6b), and the operators \([U x U']\) and \((U x U')\) can be defined as in Sections 2 and
3. Thus we may suppose that $B$ is indefinite. We further suppose that $\nu(V) = \nu(U) = \nu(U')$. Then we can choose $t_i$ so that
\[ G_\Lambda^B = \bigcup_i G_\Omega t_i V^x G_\infty = \bigcup_i G_\Omega t_i V G_\infty = \bigcup_i G_\Omega t_i U G_\infty = \bigcup_i G_\Omega t_i U' G_\infty, \]
simultaneously. Then $X_i(V^x)$ covers $X_i(U')$, and $\Gamma_i(V^x)$ is of finite index in $\Gamma_i(U')$. Thus we have the transfer map and the restriction map:
\[ T_{\Gamma_i(U')/\Gamma_i(V^x)}: H^q(\Gamma_i(V^x), \ell_i(n, \nu; A)) \rightarrow H^q(\Gamma_i(U'), \ell_i(n, \nu; A)), \]
\[ \text{res}_{\Gamma_i(U)/\Gamma_i(V)}: H^q(\Gamma_i(U), \ell_i(n, \nu; A)) \rightarrow H^q(\Gamma_i(V), \ell_i(n, \nu; A)). \]
We define
\[ T_{U'/V^x} = \bigoplus_i T_{\Gamma_i(U')/\Gamma_i(V^x)} \text{ and res}_{U/V} = \bigoplus_i \text{res}_{\Gamma_i(U)/\Gamma_i(V)} \]
and
\[ [UxU'] = T_{U'/V^x} \circ [x] \circ \text{res}_{U/V}, \quad (UxU') = T_{U'/V^x} \circ (x) \circ \text{res}_{U/V}. \]
Then by (7.4b), we have a commutative diagram:
\[ H^q(X(U), \mathcal{L}(n, \nu; A)) \equiv \bigoplus_i H^q(\Gamma_i(U), \ell_i(n, \nu; A)) \]
\[ \downarrow \quad H^q(X(U'), \mathcal{L}(n, \nu; A)) \equiv \bigoplus_i H^q(\Gamma_i(U'), \ell_i(n, \nu; A)). \]

To prove Theorems 3.1 and 4.10, we need several lemmas.

**Lemma 7.1.** Put $U(N) = \{ x \in U(N) \mid x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid d - 1 \in N_{x,N} \}$ for each ideal $N$ of $x$ outside $\Sigma^B$. Let $\ell$ be a prime ideal of $x$ with residual characteristic $\ell$ outside $\Sigma^B$, and let $e$ be the ramification index of $\ell$ over $\mathbb{Q}$. Then we have

(i) If $s > 2e/(\ell - 1)$, then $\Gamma_i(U(\ell^s))$ is torsion-free for all $i$.

(ii) For each ideal $N$ as above, if there exists a prime ideal $\ell'$ such that $\ell'$ exactly divides $N$ for $s > 2e/(\ell - 1)$, then the order of every torsion element in $\Gamma_i(N)$ is a divisor of $N_{\ell'/\ell}(\ell')^{s-1}(N_{\ell'/\ell}(\ell') - 1)$, where $\Gamma_i(N)$ stands for $\Gamma_i(U_1(N))$.

(iii) There are infinitely many square-free ideal $N$ outside $\Sigma^B$, such that $\Gamma_0(N) = \Gamma_i(U_0(N))$ is torsion-free for all $i$. We can choose $N$ so that the residual characteristic of each prime factors of $N$ is arbitrarily large.

**Proof.** We first prove the third assertion. We shall identify $R_{\ell'}$ with $M_2(\ell_\ell')$. If the image of $\gamma \in \Gamma_0(1)$ in $\Gamma_0(1) = \Gamma_0(1) / \Gamma_0(1) \cap F^\times$ is of order $n$, then...
\( \delta = \nu(\gamma)^{-1}\gamma^2 \) is of order \( n \) or \( n/2 \) in \( \Gamma_0'(1) \) and \( \nu(\delta) = 1 \). We suppose for the moment that \( \delta \neq \pm 1 \). Then \( F(\delta) < B \) is a quadratic extension of \( F \). We shall choose a prime ideal \( \ell \) which remains prime in \( F(\delta) \) and so that the \( \ell \)-adic integer ring of \( F(\delta) \) is generated by \( z \) and \( \delta \). This condition on \( \ell \) depends only on the order \( n \) but is independent of the choice of the element \( \delta \) of order \( n \), and there are infinitely many \( \ell \) with this property. If \( \delta \in \Gamma_0'(\ell) \), then as an element of \( M_2(\mathcal{O}) = \mathcal{O}_\ell \), \( \delta \) leaves the subspace \( \{ (x, 0) | x \in \mathbb{Z}/\ell \} \subset (\mathbb{Z}/\ell)^2 \) stable. This is impossible because \( z_\ell[\delta] \) is a quadratic extension of \( \mathbb{Z}/\ell \). Hence \( \Gamma_0'(\ell) \neq \delta \). Here we have implicitly chosen \( \ell \) so that \( t_\ell, \ell = 1 \). This is always possible. For each root of unity \( \xi \) such that \( [F(\xi) : F] = 2 \), we choose distinct prime ideals \( \ell_\xi \) such that \( \ell_\xi \) remains prime in \( F(\xi) \), \( \ell_\xi \notin \Sigma^B \) and \( \xi \) and \( z_\ell \) generate the \( \ell \)-adic integer ring of \( F(\xi) \). Since the number of roots of unity \( \xi \) with \( [F(\xi) : F] = 2 \) is finite, we may put \( N_0 = \prod_\xi \ell_\xi \), where \( \xi \) runs over all such roots of unity. If \( \gamma \in \Gamma_0'(N_0) \) satisfies \( \gamma^n = \varepsilon \in \mathbb{Z}^\times \), then the above argument shows that we may assume that \( n = 2 \). If \( \varepsilon = \eta^2 \) with \( \eta \in \mathbb{Z}^\times \), then \( \delta = \eta^{-1}\gamma \) is of order \( 2 \). Thus \( \delta \) is conjugate to \( (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) \) or \( (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) \) in \( M_2(\mathcal{O}) \) if one embeds \( B \) into \( M_2(\mathcal{O}) \). Since \( \nu(\delta) = 1 \), we know that \( \delta = -1 \) and \( \delta \) becomes trivial in \( \Gamma_0'(N_0) \), and \( \gamma \) also becomes trivial. If \( \varepsilon \notin (\mathbb{Z}/\ell)^2 \), then \( F(\gamma) < B \) is a quadratic extension of \( F \). The number of distinct quadratic extensions of the form \( F(\sqrt{\varepsilon}) \) for \( \varepsilon \in \mathbb{Z}^\times \) is equal to \( |\mathbb{Z}/(\sqrt{\varepsilon})^2| \). For each \( \varepsilon \in \mathbb{Z}/(\sqrt{\varepsilon})^2 \), we choose a prime ideal \( \ell_\varepsilon \) such that \( \ell_\varepsilon \notin \Sigma^B \), \( \ell_\varepsilon \) remains prime in \( F(\sqrt{\varepsilon}) \), \( \ell_\varepsilon \) is prime to \( N_0 \) and the \( \ell \)-adic integer ring of \( F(\sqrt{\varepsilon}) \) is generated by \( z_\ell \) and \( \sqrt{\varepsilon} \). Put \( N = N_0 \cdot \prod_\varepsilon \ell_\varepsilon \). We can choose \( \ell_\xi \) and \( \ell_\varepsilon \) so that their residual characteristics are arbitrarily large. If \( \varepsilon' = \eta^2 \varepsilon \) with \( \eta \in \mathbb{Z}^\times \), then \( \varepsilon_\ell + \sqrt{\varepsilon_\ell} = \eta (z_\ell + \sqrt{\varepsilon_\ell}) = z_\ell + \sqrt{\varepsilon_\ell} \). Thus the condition on \( \ell_\varepsilon \) depends only on the class of \( \varepsilon \) in \( (\mathbb{Z}/\ell^2)^2 \). Then an argument similar to the case \( \ell_\xi \) prohibits the existence of a non-trivial torsion element in \( \Gamma_0'(N) \). This proves the third assertion.

Now we shall prove the first assertion. Put

\[
X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}) \mid c \in \ell^s, \ a \equiv d \equiv 1 \mod \ell^s \right\},
\]

\[
\mathcal{X} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}) \mid a, c, d \in \ell^s \right\}.
\]

We consider the exponential map \( \exp : \mathcal{X} \to X \) and the logarithm \( \log : X \to \mathcal{X} \) defined by \( \exp(T) = \sum_{n=0}^{\infty} T^n/n! \) and \( \log(1 + T) = \sum_{n=1}^{\infty} (-1)^{n+1} T^n/n. \) Let us firstly check the \( \ell \)-adic convergence of these series. We fix a prime element \( \pi \) of \( \mathcal{O} \) and take a quadratic extension \( A = \mathcal{O}[\sqrt{\pi}] \), which is a complete valuation ring. Put \( a = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\pi} \end{pmatrix} \) as an element of \( M_2(A) \). Then we see that
\( \alpha^{-1}\mathcal{X}\alpha \subset \sqrt{\pi} \mathcal{M}_2(A) \). Thus if \( s > 2e/\ell - 1 \) (\( 2e \) is the ramification index of \( A/\mathbb{Z}_\ell \)), the convergence of \( \exp: \mathcal{X} \rightarrow X \) and \( \log: X \rightarrow \mathcal{X} \) follows from a standard argument (see e.g. [17, §3, Lemma 3]). Since
\[
\exp(\alpha^{-1}\mathcal{X}\alpha) = \alpha^{-1}\exp(\mathcal{X})\alpha \quad \text{and} \quad \log(\alpha^{-1}\mathcal{X}\alpha) = \alpha^{-1}\log(\mathcal{X})\alpha,
\]
we have well-defined \( \exp: \mathcal{X} \rightarrow X \) and \( \log: X \rightarrow \mathcal{X} \). Since \( \exp \) and \( \log \) are mutually inverse, we know that \( \mathcal{X} \equiv X \). Since \( \log(x^n) = n \log(x) \) for each integer \( n \), we know that \( X \) is torsion-free. One can always decompose
\[
G^B_A = \bigcap_i G_{Q_i}t_iU_{G_{\infty}} \quad \text{for} \ U = U(\ell^s) \quad \text{with} \ t_i \text{ satisfying } t_i,_{\ell} = 1;
\]
thus, \( \Gamma^i(U) \) can be embedded into \( X \) and hence \( \Gamma^i(U) \) is torsion-free. Note that \( \mathcal{X} = X/X \cap \mathfrak{h}_X^\times \) is also an \( \ell \)-adic Lie group and is torsion-free. Since \( \Gamma^i(U) \) is isomorphic to the image of \( \Gamma^i(U) \) in \( \mathcal{X} \), \( \Gamma^i(U) \) is also torsion-free. By Proposition 6.1, the torsion-freeness of \( \Gamma^i(U) \) for all \( i \) does not depend on the choice of \( t_i \), and hence the first assertion follows.

Finally we shall prove the second assertion. We have an exact sequence for \( U = U(\ell^s): 1 \rightarrow U_{\gamma}/U_{\gamma} \cap \mathfrak{h}_\gamma^\times \rightarrow U_{(\ell^s)}/U_{\gamma} \cap \mathfrak{h}_\gamma^\times \rightarrow (\gamma/\ell^s\gamma)^{\times} \rightarrow 1. \)
\[
\left( \begin{array}{cc}
a & b \\ c & d
\end{array} \right) \quad \text{ad}^{-1} \mod \ell^s
\]
If \( T \subset \Gamma^i_1(N) \) is a finite subgroup, \( T \) injects into \( (\gamma/\ell^s\gamma)^{\times} \) since \( U_{\gamma}/U_{\gamma} \cap \mathfrak{h}_\gamma^\times \) is torsion-free. Thus we see that \( T \) is a divisor of \( N(\ell)^{s-1}(N(\ell) - 1) \) which is the order of \( (\gamma/\ell^s\gamma)^{\times} \).

\textbf{Proof of Theorems 3.1 and 4.10.} We shall prove the two theorems simultaneously. By (2.6c), (6.3) and (7.4a), the isomorphism of Theorem 6.2 is equivariant under the abstract Hecke ring \( R(U_1(N), \Delta_1(N)) \). When \( F = \mathbb{Q} \), the assertions of Theorems 3.1 and 4.10 are well-known (see e.g. [36, III]). Thus we may assume that \( F \neq \mathbb{Q} \). Then we can find a division quaternion algebra \( B/F \) which is unramified everywhere at finite places of \( F \). We fix such a \( B \). Let \( K_0 \) be the finite extension of \( \mathbb{Q} \) as in (1.1) for this \( B \). Firstly, we suppose that the algebra \( A \) as in Theorem 3.1 contains the integer ring \( \mathfrak{o}_0 \) of \( K_0 \). Let \( L \) be the quotient field of \( A \). We put for each open subgroup \( U \) of \( \text{GL}_2(\mathfrak{i}) \),
\[
S(U) = \bigoplus_{J \subset I_0} S_{k, w, J}(U; B; C)
\]
and identify \( S(U) \) with the subspace of \( H^r(X(U), \mathcal{L}(n, v; C)) \). We take a normal open compact subgroup \( U \) of \( U_1(N) \) such that \( \Gamma^i(U) \) is torsion-free for all
Then, by using Hochschild-Serre spectral sequence [16], we know
\[(7.5) \quad H^i(X(U), \mathcal{L}(n, v; L))^{U_i(N)} \cong \left( \bigoplus_i H^i(\overline{\Gamma}_i(U), \mathcal{T}_i(n, v; L))^{U_i(N)} \right)
\cong \left( \bigoplus_j H^i(\overline{\Gamma}_j(N), \mathcal{T}_j(n, v; L)) \right),
\]
where we have written $\overline{\Gamma}_i(N)$ for $\overline{\Gamma}_i(U_i(N))$. Since on $iH_r F(N)$, $tL_i(n, v; L)$ is a multiplication of the degree $(X(U): X(U_1(N)))$, Theorem 6.3 combined with (7.5), shows that
\[\left( \bigoplus_i H^i(\overline{\Gamma}_i(N), \mathcal{T}_i(n, v; A)) \right) \otimes_A C \cong H^i(X(U_1(N)), \mathcal{L}(n, v; C)).\]

Let $H = H(A)$ be the image of $\bigoplus_i H^i(\overline{\Gamma}_i(N), \mathcal{T}_i(n, v; A))$ in $S = S(U_i(N))$ under the isomorphism of Theorem 6.2. In the same manner as in the proof of [7, 4.6], we can show that $H \otimes_A C = S$ even when $r$ is even and $n = 0$. Since $H$ is stable under the action of $R(U_1(N), \Delta_i(N))$, we know that $\mathcal{H}_k, w(N; A) \hookrightarrow \text{End}_A(H)$. Thus $\mathcal{H}_k, w(N; A)$ is a flat $A$-module of finite type. For any $A$-algebra $D$ inside $C$, $H \otimes_A D$ is a $D$-submodule of $S$ stable under $\mathcal{H}_k, w(N; D)$ and hence
\[\mathcal{H}_k, w(N; D) \hookrightarrow \text{End}_A(H \otimes_A D) \cong \text{End}_A(H) \otimes_A D\]
since $D$ is $A$-flat and $H$ is $A$-projective. Since $\mathcal{H}_k, w(N; D)$ is generated over $D$ by $T_0(\sigma)$ for all $\sigma$, we know from this fact
\[\mathcal{H}_k, w(N; D) \cong \mathcal{H}_k, w(N; A) \otimes_A D.\]

Before proving Theorem 3.1 in general, we shall prove Theorem 4.10. By the duality in Theorem 5.1, applying the above identity to $D = K_0$ and with $A$ as in Theorem 4.10 for $K = K_0$, we know that
\[S_{k, w, l}(N; M_2(F); K_0) \cong S_{k, w, l}(N; M_2(F); A) \otimes_A K_0.\]

On the other hand, by Corollary 4.5, we know that
\[S_{k, w, l}(N; M_2(F); K_0) \cong S_{k, w, l}(N; M_2(F); \Phi(v)) \otimes_{\Phi(v)} K_0.\]

In order to treat the general case where $A$ is an arbitrary $\mathcal{E}(v)$-subalgebra of a finite extension $K/\Phi(v)$, we change the notation and denote by $L$ the quotient field of $A$; so, $L \subset K$. If there is an element $f \in S_{k, w, l}(N; L)$, we can find, by the above facts, $a \in L^*$ such that $af \in S_{k, w, l}(N; A)$. This shows that
\[S_{k, w, l}(N; M_2(F); K) \cong S_{k, w, l}(N; M_2(F); L) \otimes_L K\]
\[= S_{k, w, l}(N; M_2(F); A) \otimes_A K,
\]
and thus Theorem 4.10 follows. Replacing \( H \) and \( S \) by \( S_{k, w, f}(N; A) \) and \( S_{k, w, f}(N; C) \), we see that the reasoning which proves Theorem 3.1 in the case: \( A \supset \mathfrak{a}_0 \) still works well in the general case of \( A \supset \mathfrak{a}(v) \); so, we now obtain the theorem.

Now we shall determine the structure of cohomology groups \( H^r(X(U_1(N)), \mathcal{L}(n, v; K)) \) as modules over the Hecke algebra when \( K \) is a field.

**Theorem 7.2.** Let \( B \) be a division quaternion algebra over \( F \) unramified at every finite place, and put \( r = |I_B| = \dim_C \mathcal{Z}_B \). For each field extension \( K/K_0 \) for the field \( K_0 \) as in (1.1), \( H^r(X(U_1(N)), \mathcal{L}(n, v; K)) \) is free of rank \( 2^t \) over \( \mathcal{H}_{k, w}(N; K) \) if either \( k > 2t \) or \( r \) is odd.

(Here we have used the notation of Theorem 6.2 and \( H^r(X(U_1(N)), \mathcal{L}(n, v; K)) \) means \( \mathcal{T}_1 H^r(\overline{\Gamma}_1(N), \mathcal{L}(n, v; K)) \) when \( \overline{\Gamma}_1(N) \) has non-trivial torsion elements.)

**Proof.** Let \( H = H(A) \) be as in the proof of Theorems 3.1 and 4.10. Then \( H(K_0) \) is stable under the action of \( [U_1(N) \times U_1(N)] \) for all \( x \in G_F \). Put \( C = C_{\infty}/C_{\infty}^+ \), which is isomorphic to \( \{ \pm 1 \}^{I_B} \) as a group. Then \( C \) acts on \( H(K_0) \) via \( [U_1(N) \times U_1(N)] \) for \( c \in C_{\infty} \). For each character : \( C \to \{ \pm 1 \} \) and for each subalgebra \( A \) of \( C \), we define

\[ H_\epsilon(A) = \{ m \in H(A) | m|c = \epsilon(c) m \text{ for all } c \in C \}. \]

Since \( C \) acts on the subsets of \( I_B \) transitively via \( J \to J^c \) as in (2.2a) and since the action of \( c \in C \) induces an isomorphism: \( S_{k, w, f}(N) \equiv S_{k, w, f}(U) \) by Theorem 2.2, we know that

\[ S_{k, w, f}(N; B; C) \equiv H_\epsilon(C) \equiv H_\epsilon(K_0) \otimes_{K_0} C \text{ as } \mathcal{H}_{k, w}(N; C) \text{-modules} \]

via \( f \mapsto \sum_{s \in C} \epsilon(s) \omega(f) |s \). We can define a non-degenerate pairing

\[ [ , ] : L(n^B, v^B; C) \times L(n^B, v^B; C) \to C \]

such that \( [x \gamma, y \gamma] = [x, y] \) for \( \gamma \in \overline{\Gamma}_1(N) \) (cf. [29, II], [36, 8.2], [35, (1.2a, b)]).

Then for \( f \in S_{k, w, f}(N; B; C) \) and \( g \in S_{k, w, \phi}(N; B; C) \), we define

\[ (f, g) = \sum_i \int_{X_i(U_1(N))} \left[ f_i(z), g_i(z) \right] \cdot \text{Im}(z)^{kB} d\mu(z), \]

where \( f_i = f_{i,i} \) and \( g_i = g_{i,i} \) as in (2.4b) and

\[ d\mu(z) = \text{Im}(z)^{-2tB} \cdot \prod_{\sigma \in I_B} |dz_\sigma \wedge d\bar{z}_\sigma|. \]

Now let \( \omega \) be an element of \( G_f^B \) such that \( \omega_N = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) for \( \nu \in t_N \) with \( \nu t_N = N \) and \( \omega = 1 \) for all places \( \sigma \) outside \( N \). We decompose

\[ S_{k, w, f}(N; B; C) = \bigoplus_{\psi} S_{k, w, f}(N, \psi; C), \]
where \( \psi \) runs over characters of \( \text{Cl}_f(N) \) and
\[
S_{k, w, f}(N, \psi; C) = \left\{ f \in S_{k, w, f}(N; B; C) \mid f(\langle a \rangle) = \psi(a) f \text{ for } a \in \mathbb{I}(N) \right\}.
\]
For each \( f \in S_{k, w, f}(N, \psi; C) \), regarding \( \psi \) as an idele character, we define
\[
(f|W)(x) = \tilde{\psi}(v(x)) f(x\omega).
\]
Then it is known that \( W \) gives an automorphism of \( S_{k, w, f}(N; B; C) \) (see e.g. [35, Lemma 1.3], [8, 3.9]) and if we put \( \langle f, g \rangle = (f, g|W) \), then it satisfies
\[
\langle f|h, g \rangle = \langle f, g|h \rangle \text{ for } h \in \mathbb{A}(N; C) \text{ (e.g. [14, Lemma 6.4], [15, §3]).}
\]
This shows that
\[
S_{k, w, f}(N; B; C) \equiv \text{Hom}_C(S_{k, w, f}(N; B; C), C) \text{ as an } \mathbb{A}(N; C) \text{-module.}
\]
On the other hand, \( S_{k, w, f}(N; B; C) \equiv S_{k, w, f}(N; M_2(F); C) \) as an \( \mathbb{A}(N; C) \)-module by Theorems 2.1 and 2.2, and Proposition 2.3. By Theorem 5.1, we know
\[
S_{k, w, f}(N; M_2(F); C) \equiv \text{Hom}_C(\mathbb{A}(N; C), C) \text{ as an } \mathbb{A}(N; C) \text{-module.}
\]
These facts show that, as \( \mathbb{A}(N; K_0) \)-modules,
\[
H_i(K_0) \otimes_{K_0} C \equiv S_{k, w, f}(N; C) \equiv \mathbb{A}(N; K_0) \otimes_{K_0} C.
\]
Hence we know
\[
(7.6) \quad H_i(K_0) \equiv \mathbb{A}(N; K_0) \text{ as } \mathbb{A}(N; K_0) \text{-modules.}
\]
Then we conclude the assertion of the theorem by extending the scalar field to \( K \) from \( K_0 \).

Remark 7.3. We have proved actually a little stronger result than the statement of Theorem 7.2. Namely, \( H(K) \) for any field extension \( K/K_0 \) is free of rank \( 2^r \) over \( \mathbb{A}(N; K) \), even when \( r \) is even and \( k = 2t \).

We shall record a byproduct of the proof of Theorem 7.2 (cf. [14, Lemma 6.4]):

Corollary 7.4. \( \mathbb{A}(N; \Phi(v)) \) is a Frobenius algebra over \( \Phi(v) \).

8. Comparison between cohomology groups of different weights

Let \( U \) be an open compact subgroup of \( \hat{R}^\times \) and suppose:
\[
(8.1a) \quad \text{Every prime factor of } p \text{ is unramified in } B.
\]
We suppose the following condition when we consider sheaf cohomology groups:
\[
(8.1b) \quad \bar{\Gamma}^i(U) \text{ is torsion-free for all } i.
\]
Now we shall define several morphisms between cohomology groups. Almost
all results here will be formulated for the sheaf cohomology groups $H^q(X(U), \mathcal{L}(n, v, \lambda; A))$ under the assumptions (8.1a,b) but can be naturally reformulated in a standard manner in terms of group cohomology: \( \oplus_i H^q(\Gamma(U), i^*L_i(n, v, \lambda; A)) \), and then the result will be valid without the assumption (8.1b). The interpretation is automatic and is left to the reader.

By definition, \( L(n, v, \lambda; A) \) as in Section 7 is the space of homogeneous polynomials of variables \( (X_{\sigma}, Y_{\sigma})_{\sigma \in I} \). We evaluate each polynomial \( P(X_{\sigma}, Y_{\sigma}) \in L(n, v, \lambda; A) \) at \( (X_{\sigma}, Y_{\sigma}) = (1, 0) \) for all \( \sigma \in I \), and we then have a morphism of an \( A \)-module:

\[
i: L(n, v, \lambda; A) \to A.
\]

Let \( K \) be a finite extension of \( \mathbb{Q}_p \) containing \( K_0 \) as in (1.1), and let \( \mathcal{O} \) be the \( p \)-adic integer ring of \( K \). Let \( \lambda: \mathbb{Z}_0/\mathbb{Z}_\beta \to \mathcal{O}^\times \) be a character for integers \( \beta \geq \alpha \geq 0 \). For each \( k \in \mathbb{Z}[I] \), we occasionally denote by \( \chi_k \) the character:

\[
\iota_p^\times \ni x \mapsto x^k = \prod_{\sigma \in I} (x^\sigma)^{k^\sigma} \in \mathcal{O}.
\]

This notation is consistent with the character \( \chi_k: Z(N) \to \mathbb{Z}_p^\times \) already defined in Section 3 when \( k \sim 0 \). Since \( u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) (for \( u \in U_b^a(p^\beta) \)) satisfies the congruence: \( u_p \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod p^\beta \mathbb{M}_2(\mathbb{Z}_p^\times) \) and its action on \( L(n, v, \lambda; A) \) for \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( p^{-\beta} \mathcal{O}/\mathcal{O} \) factors through the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}/p^\beta) \), we know that for \( u \in U_b^a(1) \),

\[
i(P \cdot u_p) = \lambda \chi_n(a) \nu(u_p)^v i(P) \quad \text{for} \quad P \in L(n, v, \lambda; A).
\]

Then \( i \) induces a morphism of sheaves for all \( U \subset U_b^a(1) \):

\[
i: \mathcal{L}(n, v, \lambda; A)_{/X(U)} \to \mathcal{L}(0, v, \lambda \chi_n; A)_{/X(U)}
\]

and hence we have, for \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( p^{-\beta} \mathcal{O}/\mathcal{O} \),

\[
(8.2a) \quad i_*: H^q(X(U), \mathcal{L}(n, v, \lambda; A)) \to H^q(X(U), \mathcal{L}(0, v, \lambda \chi_n; A)).
\]

Here we have abused symbols slightly. In fact, \( \lambda \chi_n \) is not necessarily a character of \( \mathbb{Z}_a/\mathbb{Z}_\beta \) with values in \( \mathcal{O}/p^\beta \mathcal{O} \), but we can let \( u \in \Delta^a_\beta \) with \( u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) act on \( L(0, v; A) \) by \( P \cdot u = \lambda \chi_n(a)(P \cdot u_p) \), where \( P \cdot u_p \) on the right-hand side denotes the original action of \( u_p \) on \( L(0, v; A) \). Each element in the center \( a \in \iota_p^\times \) with \( a \equiv 1 \mod p^\alpha \iota_p \) acts then on \( L(0, v, \lambda \chi_n; A) \) by \( \lambda(a) \chi_{n+2\alpha}(a) \); and hence \( E \) acts trivially on this module. Thus if the sheaf \( \mathcal{L}(n, v, \lambda; A) \) is well-defined on \( X(U) \), then \( \mathcal{L}(0, v, \lambda \chi_n; A)_{/X(U)} \) is also well-defined. As with \( i \) above, identifying \( L^*(0, v, \lambda \chi_n; A) \) with \( A \) (for \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( p^{-\beta} \mathcal{O}/\mathcal{O} \)), one can define another \( V_\beta^p(1) \)-morphism

\[
j: L^*(0, v, \lambda \chi_n; A) \to L^*(n, v, \lambda; A)
\]
by \( j(a) = aY^n = a \cdot \prod_{\sigma \in \Gamma} \gamma^n_{\sigma} \). This map induces, for \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( \mathcal{O}/p^{-\beta} \mathcal{O}/\mathcal{O} \),
\[
(8.2b) \quad j_*: H^q(X(U), \mathcal{L}^*(0, v, \lambda \chi_n; A)) \to H^q(X(U), \mathcal{L}^*(n, v, \lambda; A))
\]
if \( U \subset V^\alpha_\beta(1) \).

Now we take \( \omega = \omega_\beta \in G^B_f \) such that \( \omega_p = \begin{pmatrix} 0 & 1 \\ -p^\beta & 0 \end{pmatrix} \) and \( \omega_\sigma = 1 \) for all places \( \sigma \) outside \( p \). We note the identity inside \( M_{2, \mathcal{O}_p} \):
\[
\begin{pmatrix} a & b \\ p^\beta c & d \end{pmatrix} \omega_p = \begin{pmatrix} d & -c \\ -p^\beta b & a \end{pmatrix}.
\]
Let \( \phi: L(0, v, \lambda \chi_n; A) \to L^*(0, v, \lambda \chi_n; A) \) be the identity map of the underlying space \( A \). Then if \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( \mathcal{O}/p^{-\beta} \mathcal{O}/\mathcal{O} \), \( \phi(m \cdot u_p) = \phi(m)(\omega^{-1}u \omega)_p \) for \( u \in U^{\alpha}_\beta(1) \) and \( m \in L(0, v, \lambda \chi_n; A) \). Therefore, the map
\[
[\omega]: G_A \times L(0, v, \lambda \chi_n; A) \to G_A \times L^*(0, v, \lambda \chi_n; A)
\]
induces a morphism of sheaves
\[
[\omega]: \mathcal{L}(0, v, \lambda \chi_n; A)_{/X(U)} \to \mathcal{L}^*(0, v, \lambda \chi_n; A)_{/X(U^\omega)}
\]
if \( U \subset U^\alpha_\beta(1) \). Thus, if \( U \subset U^\alpha_\beta(1) \) and \( A = \mathcal{O}/p^\beta \mathcal{O} \) or \( \mathcal{O}/p^{-\beta} \mathcal{O}/\mathcal{O} \),
\[
(8.2c) \quad W = [\omega]: H^q(X(U), \mathcal{L}(0, v, \lambda \chi_n; A)) \to H^q(X(U^\omega), \mathcal{L}^*(0, v, \lambda \chi_n; A)).
\]

Let \( \delta \) be an element of \( G^B_f \) such that \( \delta_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \delta_\sigma = 1 \) for all other places \( \sigma \). Then for \( V = U^\omega \cap \delta U \delta^{-1} \), if \( U \subset \Delta^\alpha_\beta \), then \( V \subset (\Delta^\alpha_\beta)^U \) and \( V^\delta = \delta^{-1}U^\omega \delta \cap U \subset \Delta^\alpha_\beta \). We consider the map
\[
[\delta]: L^*(n, v, \lambda; A) \to L(n, v, \lambda; A)
\]
defined by \( m \mapsto \phi(m \cdot \delta) \), where \( \delta \) acts on \( L(n, v, \lambda; A) \) through the identification of the underlying space \( L(n, v, \lambda; A) = L(n, v; A) \) and \( \phi \) is induced by the identity map of the underlying space \( L(n, v; A) \). Then we see that
\[
[\delta](m \cdot u) = ([\delta](m)) \cdot u^\delta \text{ for } u \in V \ (u^\delta = \delta^{-1}u \delta).
\]
Thus this induces a morphism of sheaves
\[
[\delta]: \mathcal{L}^*(n, v, \lambda; A)_{/X(V)} \to \mathcal{L}(n, v, \lambda; A)_{/X(V^\delta)}.
\]
Therefore, we can define
\[
[U^\omega \delta U]: H^q(X(U^\omega), \mathcal{L}^*(n, v, \lambda; A)) \to H^q(X(U), \mathcal{L}(n, v, \lambda; A))
\]
by
\[
(8.2d) \quad [U^\omega \delta U] = \text{Tr}_{U/V^\delta} \circ [\delta] \circ \text{res}_{U^\omega/V}.
\]
We now suppose that
\[(8.3) \quad U_p = \mathbb{P} \times \mathbb{P} \quad \text{and} \quad U = U_p \times U_p \quad \text{for} \quad U_p = \{ x_p | x \in U \} \quad \text{and} \quad U^p = \{ x \in U | x_p = 1 \}.
\]
We put \(U^\alpha_\beta = U \cap U^\alpha_\beta(1)\) for integers \(\beta \geq \alpha \geq 0\). For each \(a \in \mathbb{T}_p\), we define \(\delta_a \in \mathcal{G}_f^B\) by \((\delta_a)_p = \left( \frac{1}{0} a \frac{1}{1} \right)\) and \((\delta_a)_\sigma = 1\) for \(\sigma\) outside \(p\). Then we see
\[(8.4) \quad (U^\alpha_\beta)^{\omega} \delta U^\alpha_\beta = \prod_a (U^\alpha_\beta)^{\omega} \delta_a \quad \text{if} \beta > 0,
\]
where \(a\) runs over a complete representative set for \(\mathbb{T}_p/p^\beta \mathbb{T}_p\). Then we shall define morphisms (for \(A = \mathcal{O}/p^\beta \mathcal{O}\) or \(p^{-\beta} \mathcal{O}/\mathcal{O}\)):
\[
\pi = \pi_\beta : H^q(X(U^\alpha_\beta), \mathcal{L}(n, v, \lambda; \chi_n; A)) \to H^q(X(U^\alpha_\beta), \mathcal{L}(n, v, \lambda; A)),
\]
\[
\iota = \iota_\beta : H^q(X(U^\alpha_\beta), \mathcal{L}(n, v, \lambda; A)) \to H^q(X(U^\alpha_\beta), \mathcal{L}(0, v, \lambda; \chi_n; A))
\]
by \(\iota = i_*\) and \(\pi = [(U^\alpha_\beta)^{\sigma} \delta U^\alpha_\beta] \circ j_* \circ W\). We fix a character of semi-group: \(\Pi(1) \ni a \mapsto \{ a^\sigma \} \in \mathcal{O}\) as in Section 3 and define operators \((U^\alpha_\beta x U^\alpha_\beta)\) for \(x \in \Delta^\alpha_\beta\) as in Section 7. Then the importance of the morphisms \(\iota\) and \(\pi\) comes from the following result:

**Theorem 8.1.** Let \(A = \mathcal{O}/p^\beta \mathcal{O}\) or \(p^{-\beta} \mathcal{O}/\mathcal{O}\). For each pair of integers \((\alpha, \beta)\) with \(\beta \geq \alpha \geq 0\) and \(\beta > 0\),
\[
\pi \circ \iota = p^{-\beta_0} \{ p^{\beta_0} \} T_0(p^{\beta}) \text{ on } H^q(X(U^\alpha_\beta), \mathcal{L}(n, v, \lambda; A)),
\]
\[
\iota \circ \pi = p^{-\beta_0} \{ p^{\beta_0} \} T_0(p^{\beta}) \text{ on } H^q(X(U^\alpha_\beta), \mathcal{L}(0, v, \lambda; \chi_n; A))
\]
where \(T_0(p^{\beta}) = (U^\alpha_\beta x U^\alpha_\beta)\) for \(x\) with \(x_p = \left( \begin{array}{cc} 1 & 0 \\ 0 & p^\beta \end{array} \right)\) and \(x_\sigma = 1\) for \(\sigma\) outside \(p\). Furthermore \(\iota\) is equivariant under the operators \((U^\alpha_\beta y U^\alpha_\beta)\) for \(y \in \Delta^\alpha_\beta\) on both the cohomology groups. These assertions are also valid for cohomology groups with compact support.

We note that \(p^{-\beta_0} \{ p^{\beta_0} \}\) is a unit in \(\mathcal{O}\), and therefore, \(\iota \circ \pi\) and \(\pi \circ \iota\) are unit multiplies of \(T_0(p^{\beta})\).

**Proof.** We shall prove the assertions only for the usual cohomology groups since the case of compact support can be handled in exactly the same manner. Let \(\phi : L(n, v, \lambda; A) \to L^*(n, v, \lambda; A)\) be the identity map of the underlying space \(L(n, v; A)\). Let \(m \mapsto m \circ \omega\) denote the action of \(\omega\) on \(L(n, 0; A)\). Identifying \(L(n, v, \lambda; A)\) and \(L^*(n, v, \lambda; A)\) with \(L(n, 0; A)\) as \(A\)-modules naturally, we define
\[
(\omega)_0 : G_A \times L(n, v, \lambda; A) \to G_A \times L^*(n, v, \lambda; A) \quad \text{by} \quad (\omega)_0(g, m) = (g \omega, \phi(m \circ \omega)).
\]
Then we can easily check that \((\omega)_0\) induces a morphism of sheaves

\[
(\omega)_0: \mathcal{L}(n, v, \lambda; A)_{/X(U_0^\alpha)} \to L^*(n, v, \lambda; A)_{/X(U_0^\alpha)}
\]

and a morphism of cohomology groups

\[
(\omega)_0: H^q(X(U_0^\alpha), \mathcal{L}(n, v, \lambda; A)) \to H^q(X((U_0^\alpha)^w), \mathcal{L}^*(n, v, \lambda; A)).
\]

Since \(\omega_r \left( \frac{x}{y} \right) = \left( \frac{y}{x} \right) \) on \((A')^2\), we have \(j(i(m)) = m \circ \omega_r\) for \(m \in L(n, v, \lambda; A)\). Then we see that

\[
j_* \circ W \circ i_* = (\omega)_0: H^q(X(U_0^\alpha), \mathcal{L}(n, v, \lambda; A)) \to H^q(X((U_0^\alpha)^w), \mathcal{L}^*(n, v, \lambda; A)).
\]

This shows that

\[
\pi \circ i = \left[ \left( U_0^\alpha \right)^w \delta U_0^\alpha \right] \circ (\omega)_0 = \left( p^{-\beta v} \{ p^{\beta v} \} \right) \left( U_0^\alpha \omega \delta U_0^\alpha \right) = \left( p^{-\beta v} \{ p^{\beta v} \} \right) T_0(p^\beta),
\]

because \(x = \omega \delta \in \Delta^\alpha_\beta\). Now we shall prove the second identity about \(i \circ \pi\). We have by definition that \(i \circ \pi = i_* \circ \left[ (U_0^\alpha)^w \delta U_0^\alpha \right] \circ j_* \circ W\). Put \(V = U_0^\alpha \cap xU_0^\alpha x^{-1}\) for \(x\) as in the theorem. Then we see that \(V = x^{-1}Vx = \delta^{-1}(U_0^\alpha)^w \delta \cap U_0^\alpha\). We write simply \(S\) for \(U_0^\alpha\). Then by (8.2d), the above expression of \(i \circ \pi\) shows that

\[
i \circ \pi = i_* \circ \operatorname{Tr}_{S/V} \circ \left[ \delta \right] \circ \operatorname{res}_{S^w/V^w} \circ j_* \circ [\omega] = \operatorname{Tr}_{S/V} \circ i_* \circ \left[ \delta \right] \circ j_* \circ \operatorname{res}_{S^w/V^w} \circ [\omega].
\]

The commutativity of \(i_*\) and \(\operatorname{Tr}\) follows from the fact that \(\Delta^\alpha_\beta \supset S \supset V^x\) and that \(i\) is a morphism of \(\Delta^\alpha_\beta\)-modules. Since \(x = \omega \delta\), we see that

\[
V = S \cap xSx^{-1} = S \cap \omega \delta \delta^{-1} \omega^{-1} = \omega(S^w \cap \delta S \delta^{-1} \omega^{-1}) = \omega V \delta \delta^{-1} \omega^{-1}.
\]

By definition, we have that \(\operatorname{res}_{S^w/V^w} \circ [\omega] = [\omega] \circ \operatorname{res}_{S/V}\). Thus we have that

\[
i \circ \pi = \operatorname{Tr}_{S/V} \circ i_* \circ \left[ \delta \right] \circ j_* \circ [\omega] \circ \operatorname{res}_{S/V}.
\]

Note that \(i(j(m) \circ \delta) = i(mY^n \circ \delta) = i(mX^n) = m\) for \(m \in L(0, v, \lambda X_n; A)\). Thus, as a morphism of sheaves, we see \(i \circ \delta \circ j\) is given by

\[
A \times L(0, v, \lambda X_n; A) \to G_A \times L^*(0, v, \lambda X_n; A) \quad (g, m) \mapsto (g \delta, \phi(m)).
\]

This shows that \(\left\{ p^{-\beta v} \right\} \right. \circ i \circ \pi = \operatorname{Tr}_{S/V} \circ (\omega \delta) \circ \operatorname{res}_{S/V} = T_0(p^\beta).\) As for the last assertion, for \(y \in \Delta^\alpha_\beta\), we see \(i \circ (SyS) = (SyS) \circ i\) because \(i\) is a morphism of \(\Delta^\alpha_\beta\)-modules.

We shall now define the ordinary part of cohomology groups. It is known (cf. [28, Propositions 4, 9 and 18]) that

\[
(8.5) \quad H^q(X(U), \mathcal{L}(n, v, \lambda; A)) \text{ is of finite type as an } \mathcal{O}-\text{module if } A \text{ is an } \mathcal{O}-\text{module of finite type}.
\]

When \(B = M_2(F)\), the space \(X(U)\) is not compact; so, we denote by
$H^q(X(U), L(n, v, \lambda; A))$ the image of $H^q(X(U), L(n, v, \lambda; A))$ inside the usual cohomology group $H^q(X(U), L(n, v, \lambda; A))$. By (8.5), the parabolic cohomology group $H^q_p(X(U), L(n, v, \lambda; A))$ is also an $\mathcal{O}$-module of finite type. Let $T_0(p^\beta)$ be the operator as in Th. 8.1. Since the $\mathcal{O}$-linear endomorphism algebra of the cohomology groups $H^q$ or $H^p$ over $X(U^p)$ with coefficients in $L(n, v, X; A)$ for $A = \mathcal{O}/p^\beta \mathcal{O}$ or $p^{-\beta} \mathcal{O}/\mathcal{O}$ is finite and of $p$-power torsion over $\mathcal{O}$, the limit

$$E = \lim_{\alpha \to \infty} T_0(p^\beta)\epsilon_{m, \alpha}$$

exists in the endomorphism algebra for a suitable choice of a positive integer $m$, and $e = E\epsilon_{m, 1}$ becomes an idempotent. This idempotent is determined independently of the choice of $m$ (cf. [10, p. 236]) and depends only on $T_0(p)$.

**Corollary 8.2.** The morphism $\iota$ in Theorem 8.1 induces an isomorphism:

$$eH^q\left(X(U^\alpha), L(n, v, \lambda; A)\right) \cong eH^q\left(X(U^\alpha), L(0, v, \lambda X_n, A)\right)$$

for $A = \mathcal{O}/p^\beta \mathcal{O}$ or $p^{-\beta} \mathcal{O}/\mathcal{O}$. The same type of assertion is also valid for parabolic cohomology groups.

**Proof.** We have a commutative diagram:

$$
\begin{array}{ccc}
H^q(X(U^\alpha), L(n, v, \lambda; A)) & \xrightarrow{\iota} & H^q(X(U^\alpha), L(0, v, \lambda X_n; A)) \\
\downarrow T & & \downarrow T \\
H^q(X(U^\alpha), L(n, v, \lambda; A)) & \xrightarrow{\iota} & H^q(X(U^\alpha), L(0, v, \lambda X_n; A)),
\end{array}
$$

where $T = \epsilon T_0(p^\beta)$ with $\epsilon = p^{-\beta v}\{p^{\beta v}\}$. Note that $\epsilon$ is a unit in $\mathcal{O}$ by definition. Therefore, if $p^m$ denotes the cardinality of the residue field of $\mathcal{O}$, then we have $\lim_{\alpha \to \infty} \epsilon^{p^m(p^m-1)} = 1$. Thus we know that for a suitable multiple $m'$ of $m$,

$$e = \lim_{\alpha \to \infty} T^{p^m(p^m-1)} = \lim_{\alpha \to \infty} T_0(p^\beta)^{p^m(p^m-1)}.$$

Thus $T$ is invertible on

$$eH^q\left(X(U^\alpha), L(n, v, \lambda; A)\right) \text{ and } H^q\left(X(U^\alpha), L(0, v, \lambda X_n; A)\right).$$

This combined with the above diagram shows the result.

**Proposition 8.3.** If $\lambda$ is a character of $Z_\alpha/Z_\beta$ for $\beta \geq \alpha \geq 0$ with $\beta > 0$, then the restriction map induces an isomorphism for each $\gamma \geq \beta$:

$$eH^q\left(X(U^\alpha), L(n, v, \lambda; A)\right) \cong eH^q\left(X(U^\gamma), L(n, v, \lambda; A)\right)$$

for $A = \mathcal{O}, \mathcal{O}/p^m\mathcal{O}, p^{-m}\mathcal{O}/\mathcal{O}$ or $K/\mathcal{O}$. The same type of assertion also holds for the parabolic cohomology groups.
Proof. Take \( x \in G_f^B \) such that \( x_p = \begin{pmatrix} 1 & 0 \\ 0 & p^{\delta} \end{pmatrix} \) with \( \delta = \gamma - \beta \) and \( x_\sigma = 1 \) for \( \sigma \) outside \( p \). Then, we claim that
\[
U_\gamma^\alpha x U_\beta^\alpha = U_\beta^\alpha x U_\beta^\alpha. 
\]
We simply write \( S \) for \( U_\gamma^\alpha \) and \( Q \) for \( U_\beta^\alpha \) and put \( V = S^x \cap S \) and \( V' = S^x \cap Q \). Then, we see that
\[
V_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S \mid b \in p^{\delta_p} \right\}, \\
V'_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q \mid c \in p^{\delta_p} \right\}.
\]
Thus \( Q_p = \bigcup_{u \bmod p^{\delta_p}} V'_p \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \) and \( S_p = \bigcup_{u \bmod p^{\delta_p}} V_p \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \). This shows that
\[
x_p^{-1} S_p x_p Q_p = \bigcup_{u \bmod p^{\delta_p}} x_p^{-1} S_p x_p \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right), \\
x_p^{-1} S_p x_p S_p = \bigcup_{u \bmod p^{\delta_p}} x_p^{-1} S_p x_p \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right),
\]
which proves the claim (8.6). We have a commutative diagram by the above proof of (8.6):
\[
\begin{array}{ccc}
H^q(X(V'), \mathcal{L}(n, v, \lambda; A)) & \xrightarrow{\text{Tr}} & H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^q(X(V), \mathcal{L}(n, v, \lambda; A)) & \xrightarrow{\text{Tr}} & H^q(X(S), \mathcal{L}(n, v, \lambda; A)). 
\end{array}
\]
This combined with (8.6) shows the commutativity of
\[
\begin{array}{ccc}
H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) & \xrightarrow{\text{res}} & H^q(X(S), \mathcal{L}(n, v, \lambda; A)) \\
\downarrow (QxQ) = T_0(p^{\delta}) & & \downarrow (SxQ) \quad (QxQ) = T_0(p^{\delta}) \\
H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) & \xrightarrow{\text{res}} & H^q(X(S), \mathcal{L}(n, v, \lambda; A)).
\end{array}
\]
We verify this as follows:
\[
\text{res}_{Q/S} \circ (SxQ) = \text{res}_{Q/S} \circ \text{Tr}_{Q/V'} \circ (x) \circ \text{res}_{S/V'^{-1}} = \text{Tr}_{S/V} \circ \text{res}_{V'/V} \circ (x) \circ \text{res}_{S/V'^{-1}} \quad \text{by (8.7)} \\
= \text{Tr}_{S/V} \circ (x) \circ \text{res}_{V'^{-1}/V^{-1}} \circ \text{res}_{S/V'-1} = (SxS).
\]
Similarly we can check that \((SxQ)\circ \text{res}_{Q/S} = (QxQ)\). Then an argument similar to that in the proof of Corollary 8.2 derives the result from (8.8).
COROLLARY 8.4. There is a Hecke operator equivariant isomorphism:

\[ \iota \circ \text{res} : eH^q(X(U^a), \mathcal{L}(n, v, \lambda; A)) \cong eH^q(X(U^\gamma), \mathcal{L}(0, v, \lambda \chi_n; A)) \]

(\(\gamma \geq \beta \geq \alpha > 0\))

for \(A = \mathcal{O}/p^\gamma \mathcal{O} \) or \(p^{-\gamma} \mathcal{O}/\mathcal{O}\) if \(Z_{\beta} \subset \text{Ker}(\lambda)\). The same type of assertion also holds for the parabolic cohomology groups.

This is a combination of Proposition 8.3 and Corollary 8.2.

Definition 8.5. Let \(U\) be an open subgroup of \(\hat{\mathbb{R}}^\times\) such that \(U = R_p^\times \times U^p\). For each \(n \in \mathbb{Z}[I] \) (\(n \geq 0\)) with \(n \sim -2v\),

\[ Y_q(n, v; U) = \lim_{\alpha} H^q(X(U^a), \mathcal{L}(n, v; K/\mathcal{O})) \]

\[ \cong \lim_{\alpha} \left( \bigoplus_i H^q(\tilde{\Gamma}^i(U^a), 'L(n, v; K/\mathcal{O})) \right), \]

\[ Y_q^{\text{ord}}(n, v; U) = \lim_{\alpha} eH^q(X(U^a), \mathcal{L}(n, v; K/\mathcal{O})) \]

\[ \cong \lim_{\alpha} e \left( \bigoplus_i H^q(\tilde{\Gamma}^i(U^a), 'L(n, v; K/\mathcal{O})) \right), \]

\[ Y_q(0, v, \chi_n; U) = \lim_{\alpha} H^q(X(U^a), \mathcal{L}(0, v, \chi_n; p^{-\alpha} \mathcal{O}/\mathcal{O})) \]

\[ \cong \lim_{\alpha} \left( \bigoplus_i H^q(\tilde{\Gamma}^i(U^a), 'L(0, v, \chi_n; p^{-\alpha} \mathcal{O}/\mathcal{O})) \right), \]

\[ Y_q^{\text{ord}}(0, v, \chi_n; U) = \lim_{\alpha} eH^q(X(U^a), \mathcal{L}(0, v, \chi_n; p^{-\alpha} \mathcal{O}/\mathcal{O})) \]

\[ \cong \lim_{\alpha} e \left( \bigoplus_i H^q(\tilde{\Gamma}^i(U^a), 'L(0, v, \chi_n; p^{-\alpha} \mathcal{O}/\mathcal{O})) \right), \]

where the injective limit is taken relative to the restriction maps. When \(U = U^a(\mathcal{N})\), we write \(Y_q(n, v; N)\) and \(Y_q^{\text{ord}}(n, v; N)\) etc. for these modules. Since \(v(U^a) = v(U)\) for all \(\alpha\), if \(B\) is indefinite, we can choose \(t_i \in G_F^B\) so that \(G_A^B = \prod_{i=1}^{h} G_{Q_{i}}^B U^a_G^B\) independently of \(\alpha\), and thus in this case, we can interchange \(\lim_{\alpha}\) and \(\bigoplus_i\) in the above definition; but when \(B\) is definite, one cannot choose \(t_i\) independently of \(\alpha\), and the order of \(\lim_{\alpha}\) and \(\bigoplus_i\) must be as above. The terms on the extreme right of the above definition work well without assuming (8.1b). On these modules \(Y_q(n, v; N)\) and \(Y_q^{\text{ord}}(0, v, \chi_n^i; N)\), Hecke operators \(T_0(n)\) and \(T_0(n, n)\) naturally act because the restriction maps are compatible with \(T_0(n)\) and \(T_0(n, n)\) (cf. (2.9a, b) and (3.5)).
Theorem 8.6. Let $U$ be an open subgroup of $\hat{R}^\times$ such that $U = R^\times_p \times U_p$. If $n \sim n' \sim -2v$ and $n \geq n' \geq 0$, then there is a canonical isomorphism:

\[ \gamma_q^{\text{ord}}(n, v; U) \cong \gamma_q^{\text{ord}}(0, v, \chi_n; U) \cong \gamma_q^{\text{ord}}(0, v, \chi_{n'}; U) \cong \gamma_q^{\text{ord}}(n', v; U). \]

When $U = U_1(N)$ for an ideal $N$ prime to $p$, this isomorphism is equivariant under Hecke operators $T_0(n)$ and $T_0(n, n)$ for all $n$.

By this theorem, the module $\gamma_q^{\text{ord}}(n, v; U)$ depends only on $v \mod \mathbb{Z} \cdot t$, and thus we write it as $\gamma_q^{\text{ord}}(v; U)$ and $\gamma_q^{\text{ord}}(v; N)$ when $U = U_1(N)$.

Proof. By definition, we have a commutative diagram: for all $\beta \geq \alpha > 0$,

\[ H^q(X(U_\alpha^a), \mathcal{L}(n, v; p^{-\alpha}O/O)) \xrightarrow{\text{res}} H^q(X(U_\beta^a), \mathcal{L}(n, v; p^{-\beta}O/O)) \]

\[ \xrightarrow{i_\alpha} \]

\[ H^q(X(U_\alpha^a), \mathcal{L}(0, v, \chi_n; p^{-\alpha}O/O)) \xrightarrow{\text{res}} H^q(X(U_\beta^a), \mathcal{L}(0, v, \chi_n; p^{-\beta}O/O)). \]

Thus we see that

\[ \gamma_q^{\text{ord}}(n, v; U) = \lim_{\alpha \to \beta} eH^q(X(U_\alpha^a), \mathcal{L}(n, v; K/O)) \]

\[ = \lim_{\alpha \to \beta} eH^q(X(U_\alpha^a), \mathcal{L}(n, v; p^{-\beta}O/O)) \]

\[ = \lim_{\alpha \to \beta} eH^q(X(U_\alpha^a), \mathcal{L}(n, v; p^{-\alpha}O/O)) \]

\[ \equiv \lim_{\alpha \to \beta} eH^q(X(U_\alpha^a), \mathcal{L}(0, v, \chi_n; p^{-\alpha}O/O)) \quad \text{(by Cor. 8.2)} \]

\[ = \gamma_q^{\text{ord}}(0, v, \chi_n; U). \]

As $\Delta_\alpha^a$-modules, $L(0, v, \chi_n; p^{-\alpha}O/O)$ and $L(0, v, \chi_n; p^{-\alpha}O/O)$ are the same and are equal to $L(0, v, \chi_n; p^{-\alpha}O/O)$. Thus we know that

\[ \gamma_q^{\text{ord}}(0, v, \chi_n; U) = \gamma_q^{\text{ord}}(0, v, \chi_{n'}; U) \equiv \gamma_q^{\text{ord}}(n', v; U). \]

The equivariance under Hecke operators follows from that of $i_\alpha$ in Theorem 8.1. The maps

\[ i_\beta \circ \text{res}: H^q(X(U_\alpha^a), \mathcal{L}(n, v; p^{-\beta}O/O)) \to H^q(X(U_\alpha^a), \mathcal{L}(0, v, \chi_n; p^{-\beta}O/O))\]

are compatible with the restriction map $\text{res}_{U_\beta^a/U_\gamma^a}$ for any $\gamma > \beta$. Thus we can take the limit

\[ \tau = \lim_{\beta \to \gamma} i_\beta \circ \text{res}: H^r(X(U_\alpha^a), \mathcal{L}(n, v; K/O)) \to \gamma_r(0, v, \chi_n; U) \]

for $r = |I_B| = \dim C\mathcal{Z}_B$. 

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The following result is essentially due to Shimura [30]:

**Theorem 8.7.** Suppose that \( r = \vert I_B \vert = 0 \) or 1. Then the kernel of \( \iota \) in (8.10) has only finitely many elements.

**Proof.** Let \( \Phi = \Gamma_i(U_a^\alpha) \) in the case of \( r = 1 \) and \( \Phi = (U_a^\alpha)_p \) in the case of \( r = 0 \). First we shall show that if an \( \mathcal{O} \)-submodule \( V \) in \( \text{Ker}(i)( \subset L(n, \nu; K/\mathcal{O}) \) is stable under the action of \( \Phi \), then we can find a positive integer \( \gamma \) so that \( p^\gamma V = 0 \). In fact, by the strong approximation theorem, the closure of \( \Phi \) in \( U_p \) is a \( p \)-adic Lie group containing \( \text{SL}_2(\mathbb{Z}_p) \cap U_a^\alpha \) if \( r = 1 \). Thus the \( \Phi \)-module \( L(n, \nu; K) \) is absolutely simple. Let \( M \) be the subalgebra of \( \text{End}_K(L(n, \nu; K)) \) generated over \( \mathcal{O} \) by the action of \( \Phi \). Then the simplicity of the \( \Phi \)-module \( L(n, \nu; K) \) shows that \( M \otimes \mathcal{O} K \) coincides with \( \text{End}_K(L(n, \nu; K)) \). Thus there is an element \( E_j \in M \) such that the coefficient of \( E_j(\sum_{0 \leq i \leq n} a_i X^i Y^{n-i}) \) in \( X^n = \prod_{a} X_a^\alpha \) is equal to \( p^\gamma a_j \) for all \( \sum_{i} a_i X^i Y^{n-i} \in L(n, \nu; K) \) for each \( 0 \leq j < n \). If \( P = \sum_{i} a_i X^i Y^{n-i} \in L(n, \nu; K/\mathcal{O}) \) is contained in \( V \), then \( E_j \cdot P \in V \) by the stability of \( V \) under \( M \). Since \( 0 = i(E_j \cdot P) = p^\gamma a_j \), we know that \( p^\gamma P = 0 \) and hence \( p^\gamma V = 0 \). Especially, the order of \( V \) is finite.

Next we shall deal with the proof in the case of \( r = 0 \). Let \( S \) be the space of functions \( f: G_A^B \to L(n, \nu; K/\mathcal{O}) \) satisfying \( f(axu) = f(x)u_p \) for \( u \in U_a^\alpha G_{\infty}^B \) and \( a \in G_{\infty}^B \). Then \( S \cong H^0(X(U_a^\alpha), L(n, \nu; K/\mathcal{O})) \). If \( i(f(x)) = 0 \) for all \( x \in G_A^B \) (\( f \in S \)) (this is equivalent to supposing that \( f \in \text{Ker}(i) \)), then \( f \) must have values in \( \text{Ker}(i) \) in \( L(n, \nu; K/\mathcal{O}) \). Since \( f(xu) = f(x)u_p \) for \( u \in U_a^\alpha \), the subspace in \( \text{Ker}(i) \) generated by the values \( f(x) \) for all \( x \in G_A^B \) is stable under \( \Phi \). As already seen, we then know that \( p^\gamma f = 0 \) for a positive \( \gamma \) independent of \( f \). This shows the result in the case of \( r = 0 \).

Now we suppose that \( r = 1 \). In this case, we can take the decomposition \( G_A^B = \coprod_{i=1}^m G_{Q_i}^B \beta_i G_{\infty}^B \) independently of \( \beta \). For simplicity, we hereafter write \( \Phi \) for \( \Gamma_i(U_a^\alpha) \) for a fixed \( i \). Put

\[
\Gamma_\beta = \{ \gamma \in \Gamma_i(U_a^\alpha) \mid \gamma \equiv 1 \mod \text{p}^\beta R_p \}, \quad \overline{\Gamma_\beta} = \Gamma_\beta / \Gamma_\beta \cap F^\times.
\]

Then \( H^1(\overline{\Gamma_\beta}, L(n, \nu; p^{-\beta} \mathcal{O}/\mathcal{O})) \equiv \text{Hom}(\overline{\Gamma_\beta}, L(n, \nu; p^{-\beta} \mathcal{O}/\mathcal{O})) \). By virtue of a theorem of Shimura, which is given in [26, Th. 3.1.3], the injective limit of the restriction maps

\[
I: H^1(\overline{\Phi}, L(n, \nu; K/\mathcal{O})) \equiv \lim_{\beta} H^1(\Phi, L(n, \nu; p^{-\beta} \mathcal{O}/\mathcal{O}))
\]

\[
\to \lim_{\beta} \text{Hom}_{\phi}(\overline{\Gamma_\beta}, L(n, \nu; p^{-\beta} \mathcal{O}/\mathcal{O}))
\]

is known to have a finite kernel. On Ohta’s article [26], he made an assumption
that \( B \) is a division algebra to assure (in [26, p. 27]) the vanishing of \( H^1(\Delta, L(n, v; \mathbb{C})) \) for a congruence subgroup \( \Delta \) of \( (B \otimes_F F') \) for totally real fields \( F' \neq \mathbb{Q} \) by the result of Matsushima and Shimura [22, Th. 7.1]. However, by works of Harder ([5] and [6]), this vanishing has been proved even for \( M_2(F') \) for \( F' \neq \mathbb{Q} \). Thus, this division assumption can be eliminated without affecting the original proof, and the fact: \( |\text{Ker}(I)| < \infty \) is valid even for \( B = M_2(\mathbb{Q}) \). For \( \xi \in \text{Hom}_F(\tilde{\Gamma}_B, L(n, v; p^{-\beta}\mathcal{O}/\mathcal{O})) \), we know that \( \xi(u\delta u^{-1}) = u\xi(\delta) \) for every \( u \in \Phi \) and \( \delta \in \tilde{\Gamma}_B \). If \( i \circ \xi = 0 \), then for all \( u \in \Phi \), \( i(u\xi(\delta)) = 0 \). Thus the value of \( \xi \) is contained in a \( \Phi \)-submodule of \( \text{Ker}(i) \) in \( L(n, v; K/\mathcal{O}) \). Therefore \( \text{Ker}(i) \subset p^{-\gamma}\text{Ker}(I) \) has only finitely many elements.

9. Controllability of \( \gamma^{\text{ord}}(v; U) \)

By definition, we can identify for each \( \mathcal{O} \)-module \( A \), \( H^0(X(U), \mathcal{L}(n, v, \lambda; A)) \) with the space of functions \( f: G^B_A \rightarrow L(n, v, \lambda; A) \) satisfying \( f(axu) = f(x)u_p \) for all \( a \in G^B_A \) and \( u \in UC^\infty_\mathcal{O} \). Thus, if we define for each normal subgroup \( V \) of \( U \), the action of \( U/V \) on \( H^0(X(V), \mathcal{L}(n, v, \lambda; A)) \) by

\[
f[[u]](x) = f(xu) \cdot u_p^{-1},
\]

then we have

\[
(9.1) \quad H^0(U/V, H^0(X(V), \mathcal{L}(n, v, \lambda; A))) = H^0(X(U), \mathcal{L}(n, v, \lambda; A)).
\]

We shall generalize this controllability to general cohomology groups of dimension 1. Thus we may assume that \( B \) is indefinite. We fix an open compact subgroup \( U \) of \( \tilde{\mathbb{R}}^\times \) with \( U = R^\times_p \times U^p \). Then we choose the decomposition independently of \( \alpha \) and \( \beta \):

\[
G^B_A = \prod_{i=1}^{h} G^B_{Q_i} U^\alpha_{Q_i} G^B_{\infty+} \quad \text{with } t_i \in G^B_f \quad \text{and} \quad t_{i,p} = 1.
\]

Fix integers \( \beta \geq \alpha > 0 \) and one index \( i (= 1, \ldots, h) \), and write simply \( \Gamma = \Gamma_B^\alpha \) for \( \tilde{\Gamma}^i(U^\alpha_{Q_i}) \) and \( M \) for \( {}^iL(n, v, \lambda; A) \) (for a fixed \( \mathcal{O} \)-module \( A \)). For a general group \( \Phi \) and a left \( \Phi \)-module \( X \), let \( C^q(\Phi, X) \) be the space of all functions on \( \Phi^{q+1} \) with values in \( X \). We shall define as usual the coboundary operator \( \delta = \delta^q: C^q(\Phi, X) \rightarrow C^{q+1}(\Phi, X) \) by

\[
(9.2) \quad \delta f(\gamma_0, \ldots, \gamma_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(\gamma_0, \ldots, \delta_i, \ldots, \gamma_{q+1}).
\]
Then
\[ H^q(\Phi, X) = \operatorname{Ker} \left( \delta_q : C^q(\Phi, X)^\Phi \to C^{q+1}(\Phi, X)^\Phi \right) / \operatorname{Im} \left( \delta_{q-1} : C^{q-1}(\Phi, X)^\Phi \to C^q(\Phi, X)^\Phi \right), \]
where \( \Phi \) acts on \( f \in C^q(\Phi, X) \) by \( (f \cdot \gamma)(\gamma_0, \ldots, \gamma_q) = \gamma^{-1} f(\gamma\gamma_0, \ldots, \gamma\gamma_q) \).

Now we write down explicitly the action of \([U^*_\beta x U^*_\beta]\) and \((U^*_\beta x U^*_\beta)\) on the cohomology group \(H^q(\Gamma, M)\) by using homogeneous cochains. For each \( x \in G_Q \), we can decompose \( \Gamma x \Gamma = \coprod_i \Gamma x_i \) as a disjoint union of finitely many left cosets. Write for each \( \gamma \in \Gamma \), \( x_i\gamma = \gamma(i)x_{\gamma(i)} \) for some index \( \gamma(i) (= 1, \ldots, h) \) and \( \gamma(i) \in \Gamma \). Then
\[ x_i^{-1}\gamma(i) = \gamma x_{\gamma(i)}^{-1}. \]

Then we shall define the action of \( \Gamma x \Gamma \) on \( C^q(\Gamma, M) \) by
\[
(9.3) \quad f \big| [\Gamma x \Gamma] \left( \gamma_0, \ldots, \gamma_q \right) = \sum_i x_i^{-1} \cdot f \left( \gamma_0^{(i)}, \ldots, \gamma_q^{(i)} \right), \quad \text{and} \quad f \big| (\Gamma x \Gamma) \left( \gamma_0, \ldots, \gamma_q \right) = \sum_i \left( \{ v(x_i)^{-v} \} v(x_i)^v \right) \left( x_i^{-1} \circ f \left( \gamma_0^{(i)}, \ldots, \gamma_q^{(i)} \right) \right),
\]
where we have implicitly supposed that \( x \) is contained in \( (\Delta^*_\beta) \) to have its action on \( M \) and in the second formula, we have let \( x_i^{-1} \) act on \( f(\gamma^{(i)}_0, \ldots, \gamma^{(i)}_q) \) as an element of \( L(n, 0, \lambda; A) \). Note that \( \{ v(x_i)^{-v} \} v(x_i)^v \) is a unit in \( \mathcal{O} \), and hence, the operator \( (\Gamma x \Gamma) \) is well defined. Actually, the above definition of the action of \( \Gamma x \Gamma \) depends on the choice of the representatives \( \{ x_i \} \). We shall see later that the action induced on the cohomology groups is independent of the choice of \( \{ x_i \} \); so, for a moment, we fix the decomposition: \( \Gamma x \Gamma = \coprod_i \Gamma x_i \). Now we have by the definition:
\[
(9.4a) \quad \delta(f \big| [\Gamma x \Gamma]) = (\delta f) \big| [\Gamma x \Gamma], \quad \delta(f \big| (\Gamma x \Gamma)) = (\delta f) \big| (\Gamma x \Gamma).
\]
If \( f \in C^q(\Gamma, M) \) is \( \Gamma \)-invariant, then
\[
(9.4b) \quad f \big| [\Gamma x \Gamma] \left( \gamma\gamma_0, \ldots, \gamma\gamma_q \right) = \sum_i x_i^{-1} f \left( \gamma(i)\gamma_0^{(i)}, \ldots, \gamma(i)\gamma_q^{(i)} \right) = \sum_i x_i^{-1} x_i^{(i)} f \left( \gamma_0^{(i)}, \ldots, \gamma_q^{(i)} \right) = \sum_i \gamma x_i^{-1} f \left( \gamma_0^{(i)}, \ldots, \gamma_q^{(i)} \right) \quad \text{by (\ast)}
\]
\[ = \gamma f \big| [\Gamma x \Gamma] \left( \gamma_0, \ldots, \gamma_q \right) \].

Therefore, \( (\Gamma x \Gamma) \) and \( [\Gamma x \Gamma] \) give an operator on \( C(\Gamma, M) = \bigoplus_q C^q(\Gamma, M) \) compatible with the action of \( \Gamma \) and coboundary operator \( \delta \), and hence, they act naturally on cohomology groups. We shall now show the independence of the operator \( [\Gamma x \Gamma] \) and \( (\Gamma x \Gamma) \) on \( H^q(\Gamma, M) \) from the choice of the decomposition:
\[ \Gamma x \Gamma = \bigsqcup_i \Gamma x_i. \] Since the proof is the same for (\( \Gamma x \Gamma \)) and \([\Gamma x \Gamma] \), we only deal with \([\Gamma x \Gamma] \). Let \( \Gamma' \) be a subgroup of finite index of \( \Gamma \). We decompose \( \Gamma = \bigsqcup_i \Gamma' x_i \) and define, similarly to \( \Gamma x \Gamma \), an operator \( T_{\Gamma'/\Gamma}: C^q(\Gamma', M) \to C^q(\Gamma, M) \) by
\[
 f|T_{\Gamma'/\Gamma}(\gamma_0, \ldots, \gamma_q) = \sum_i x_i^{-1} f(\gamma_i^{(i)}, \ldots, \gamma_q^{(i)}),
\]
where for each \( \gamma \in \gamma \), \( \gamma^{(i)} \in \Gamma' \) is defined by (\( \ast \)). Then it is known by Eckmann [4, Th. 7] that \( T_{\Gamma'/\Gamma}: H^q(\Gamma', M) \to H^q(\Gamma, M) \) is defined independently of the decomposition \( \Gamma = \bigsqcup_i \Gamma' x_i \). Writing \( \Gamma' \) for \( \Gamma \cap x^{-1} \Gamma x \) and \( \Gamma'' \) for \( x \Gamma' x^{-1} \), we see easily that
\[
(9.4c) \quad (\Gamma x \Gamma) = \text{res}_{\Gamma'/\Gamma} \circ [x] \circ T_{\Gamma'/\Gamma}, \quad (\Gamma x \Gamma) = \text{res}_{\Gamma'/\Gamma} \circ (x) \circ T_{\Gamma'/\Gamma},
\]
where
\[
 f|T_{\Gamma'/\Gamma}(\gamma_0, \ldots, \gamma_q) = x^{-1} f(x \gamma_0 x^{-1}, \ldots, x \gamma_q x^{-1}),
\]
\[
 f|T_{\Gamma'/\Gamma}(\gamma_0, \ldots, \gamma_q) = \left( \left( v(x)^{-v} v(x)^v \right) x^{-1} \circ f(x \gamma_0 x^{-1}, \ldots, x \gamma_q x^{-1}) \right).
\]
Thus \( [\Gamma x \Gamma] \) and \( (\Gamma x \Gamma) \) are also independent of the decomposition. Similarly to the above argument, we can define the left action of double coset \( \Gamma x \Gamma \) on the homology group \( H_q(\Gamma, M) \) by using the left coset decomposition \( \Gamma x \Gamma = \bigsqcup_i \Gamma x_i \).

Especially, on \( H_0(\Gamma, M) = M/DM \) \((DM = \sum_{\gamma \in \Gamma}(\gamma - 1)M), \Gamma x \Gamma \) acts by \( m \mapsto \sum_i x_i^{-1} \cdot m \).

Now we recall briefly the construction in [16] of the Hochschild-Serre spectral sequence in order to study the action of the double coset on the spectral sequence. We write \( \Gamma = \Gamma_{\beta}^\alpha \) and \( \Gamma' = \Gamma_{\beta}^\alpha \) for \( \beta \geq \alpha' \geq \alpha \geq 0 \) (thus \( \Gamma' \) is a normal subgroup of \( \Gamma \)). Put \( C = \sum_q C^q(\Gamma, M), \Gamma' = \sum_q C^q(\Gamma, M)^{\Gamma'} \), \( L^p,q = C^p(\Pi, C^q(\Gamma, M)^{\Gamma'}) \) for \( \Pi = \Gamma' \), \( L = \sum_{p,q} L^p,q \), \( L^q_l = \sum_{p \geq 1} L^p,q \), \( L_i = \sum_{q=0}^{\infty} L^q_i \). Then \( L_i \) gives a filtration of \( L \). On \( L^p,q \), we define a differential operator \( \delta_{\Pi}: L^p,q \to L^{p+1,q} \) relative to \( \Pi \) as in (9.2), and for \( f \in L^p,q \) put \( \delta_{\Gamma}(f)(\pi_0, \ldots, \pi_p) = \delta_{\Gamma}(f(\pi_0, \ldots, \pi_p)) \) for \( \pi_i \in \Pi \) by applying the coboundary operator relative to \( \Gamma \) to the value \( f(\pi_0, \ldots, \pi_p) \). Define \( \delta = \delta_{\Pi} + (-1)^p \delta_{\Pi} \) on \( L^p,q \). Then \( \delta \) gives a differential operator preserving the filtration (cf. [16, II]).

Let \( x \in G_F^B \) be an element such that \( x_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \) and \( x_\sigma = 1 \) for \( \sigma \) outside \( p \). Then, for sufficiently large \( m \) divisible by the order of \( F^\times \setminus F_A^\times / \nu(U) F^\times + \), we have
\[
 t_i x^m \in \gamma t_i \left( U_\beta^\alpha \right) G^B_{\infty} \quad \text{with} \quad \gamma \in G_Q^B.
\]
This follows from the strong approximation theorem. Next, we can choose the decomposition \( \Gamma' \gamma \Gamma' = \bigsqcup_i \Gamma' \gamma_i \Gamma' \). Then we have another decomposition \( \Gamma \gamma \Gamma = \bigsqcup_i \Gamma \gamma i \) for the same representatives \( \gamma_i \). This fact follows from [31, (2.19.3)] (see also Lemma 4.7 in the text). Then by (9.4b), \( (\gamma \Gamma) \) preserves \( C^q(\Gamma, M)^{\Gamma'} \), the
filtration \( \{ L_i \} \), and the double filtration \( \{ L^i_j \} \). Moreover, by (9.4a), \((\Gamma \gamma \Gamma)\) commutes with \(\delta_\Gamma\) and by definition commutes with \(\delta_\Pi\). Thus, by the construction of the spectral sequence of Hochschild and Serre [16, 1.7], the operator \((\Gamma \gamma \Gamma)\) gives an endomorphism of the spectral sequence:

\[
H^0(\Gamma / \Gamma', H^q(\Gamma', M)) \Rightarrow H^n(\Gamma, M).
\]

That is, \((\Gamma \gamma \Gamma)\) is compatible with the filtration of each term of (9.5) and commutes with all the differential maps of (9.5). Especially, \((\Gamma \gamma \Gamma)\) acts on \(H^q(\Gamma', M)\) by \((\Gamma' \gamma \Gamma')\), and this action gives an endomorphism of \(H^q(\Pi, H^q(\Gamma', M))\). By [31, (2.19.3)] (or else, as already seen in §7 in the text), the action of \((\Gamma \gamma \Gamma)\) coincides with \(T_0(p^m)\) as in Theorem 8.1. When \(M = \mathcal{I}L(n, v, \lambda; A)\) with \(A = p^{-j}C/\mathcal{O} = \mathcal{O}/p^j\mathcal{O}\), \(p^{-j}C\) is a finite power of \((\Gamma \gamma \Gamma)\) thus gives the idempotent \(e\) as in Section 8 on each \(H^n(\Gamma / \Gamma', H^q(\Gamma', M))\) or \(H^n(\Gamma, M)\). Therefore the idempotent \(e\) gives the endomorphism of the spectral sequence (9.5) for \(A\) as above. By taking the injective limit relative to \(j\), this fact is also true for \(A = k/\mathcal{O}\). Thus one knows:

**Theorem 9.1.** Suppose that \(r = |I_\beta| > 0\). Then, for the idempotent \(e\) associated with \(T_0(p)\), (9.5) gives the following spectral sequence for \(A = \mathcal{O}/p^u\mathcal{O}, p^{-u}\mathcal{O}/\mathcal{O}\) and \(K/\mathcal{O}\):

\[
H^0(\Gamma / \Gamma', eH^0(\Gamma', L(n, v, \lambda; A))) \Rightarrow eH^0(\Gamma, L(n, v, \lambda; A))
\]

for each \(\beta \geq \alpha' \geq \alpha \geq 0\) with \(\beta > 0\).

**Lemma 9.2.** Suppose that \(r = |I_\beta| > 0\). Then the idempotent \(e\) annihilates \(H^0(\Gamma, M)\) and \(H_0(\Gamma, M)\) for \(M = \mathcal{I}L(n, v, \lambda; A)\) with \(A = \mathcal{O}, \mathcal{O}/p^u\mathcal{O}, p^{-u}\mathcal{O}/\mathcal{O}\) and \(K/\mathcal{O}\).

**Proof.** We shall prove the assertion only for \(H^0(\Gamma, M)\) since the other case can be treated similarly. We may assume that \(A = \mathcal{O}/p^u\mathcal{O}\). For any given \(\rho \geq \mu\), by making \(m\) large, we can find \(\gamma_u \in G_{Q}^\beta\) for each \(u \in \mathcal{I}_p\) so that \(t^x U(p^m) = \gamma_u t^x U(p^\rho)G_{\infty}^\beta\), where \(U(p^\rho) = \{ u \in U | u_p \equiv 1 \text{ mod } p^\rho R_p \}\). Then \(\gamma_u \equiv \left( \begin{array}{c} 1 \\ \rho^m \\ 1 \end{array} \right) \text{ mod } p^\rho R_p\) and \(\Gamma \gamma \Gamma = \Pi u \text{ mod } p^\rho\) \(\Gamma \gamma_u\) by (8.6). Thus, for \(P = \sum_{0 \leq j \leq n} a_j X^j Y^{n-j}\) in \(L(n, v, \lambda; A)\),

\[
P(\Gamma \gamma \Gamma) = \left( \left( \nu(\gamma)^{-v} \nu(\gamma)^v \right) a_n \cdot \sum_{0 \leq j \leq n} \left( \begin{array}{c} n \\ j \end{array} \right) X^{n-j} Y^j \right) \sum_{u \in \mathcal{I}^u} u^j \left( \begin{array}{c} n \\ j \end{array} \right) \prod_{\sigma \in I} \left( \begin{array}{c} n_\sigma \\ j_\sigma \end{array} \right).
\]
For sufficiently large $m$, we always have $\Sigma_{u \in z/p^m} u^i = 0 \mod p^\mu$, and the lemma follows.

**Corollary 9.3.** Let $M = I^*L(n, v, \lambda; A)$ for $A = \mathcal{O}/p^n\mathcal{O}$, $p^{-\mu}\mathcal{O}/\mathcal{O}$ and $K/\mathcal{O}$ and assume that $r = |I_\mu| > 0$. Suppose that $H^i(\Gamma_\mu/\Gamma_\alpha, eH^i(\Gamma_\alpha, M)) = 0$ for $0 < i \leq q$ and $j \leq q - 1$ ($\beta \geq \alpha' \geq \alpha \geq 0$ and $\beta > 0$). Then the restriction map induces an isomorphism:

$$eH^q(\Gamma', M) / \Gamma \cong eH^q(\Gamma, M)$$

for $\Gamma = \Gamma_\beta$ and $\Gamma' = \Gamma_\alpha$.

**Proof.** Put $E_2^{i,j} = eH^i(\Pi, H^j(\Gamma', M))$ and $E^q = eH^q(\Gamma, M)$. By Theorem 9.1, we have the spectral sequence: $E_2^{i,j} \Rightarrow E^q$. We have a canonical filtration:

$$E^q = E_0^q \supset \cdots \supset E_{q-1}^q \supset 0$$

with $E_i^q / E_{i+1}^q \cong E_{\infty}^i, q-i$. We have differentials $d_k^{i,j}: E_k^{i,j} \to E_k^{i+k, j-k+1}$, and by definition

$$(*) \quad E_k^{i,j+1} = \text{Ker}(d_k^{i,j}) / \text{Im}(d_k^{i-k, j+k-1}).$$

Since $E_2^{i,j} = 0$ for $i < 0$ or $j < 0$ by Lemma 9.2, we know that $E_k^{i,j} = 0$ for $i < 0$ or $j < 0$. Thus if $i - k < 0$ and $j - k + 1 \leq 0$ (i.e. $k > i$ and $k > j + 1$), then $E_k^{i,j} = 0$. Thus $E^q / E_{q-1}^q \cong E_k^{i,j}$ for $k \geq i + 1$ and $k \geq q + 1$. If $0 < i \leq q$ and $j \leq q - 1$, by assumption

$$(**) \quad E_2^{i,j} = H^i(\Pi, eH^j(\Gamma', M)) \cong eH^i(\Gamma, H^j(\Gamma', M)) = 0.$$
(Strictly speaking, when \( U = U_1(N) \cap V(L) \), \( Z_\alpha(NL)/Z_\alpha(NL) \) acts on the cohomology group, but we know that if \( \alpha \geq 0 \), \( W_\alpha(NL) = W_\alpha(N) \). If \( \alpha \) is sufficiently large so that \( \text{Ker}(\lambda) \supset Z_\alpha \), then as \( U_\alpha \)-module, \( L(n, v, \lambda; A) = L(n, v; A) \). Thus we know that

\[
\varphi^\text{ord}_q(v; U) \equiv \lim_{\alpha \to \infty} eH^q(X(U_\alpha), \mathcal{L}(n, v, \lambda; K/\mathcal{O})) = \lim_{\alpha \to \infty} eH^q(X(U_\alpha), \mathcal{L}(n, v; K/\mathcal{O})).
\]

However, the action of \( \overline{Z} \) on the extreme right module and that of the middle term is different. We let \( \overline{Z} \) act on \( \varphi^\text{ord}_q(v; U) \) by identifying it with \( \lim_{\alpha \to \infty} eH^q(X(U_\alpha), \mathcal{L}(n, v; K/\mathcal{O})) \) via the character \( \mathbb{I}(N) \equiv a \mapsto T(a, a) \). This is independent of the choice of \( n \) with \( n \sim -2v \) by Theorem 8.6. We write this standard action on \( \varphi^\text{ord}_q(v; U) \) as \( \xi \mapsto \xi(\langle z \rangle) (z \in \overline{Z}) \), and the action induced by the middle term as \( \xi \mapsto \xi(\langle z \rangle)_n, v, \lambda \). The latter action factors through the finite group \( \overline{Z}_\alpha/\overline{Z}_\beta \) on \( eH^q(X(U_\alpha), \mathcal{L}(n, v, \lambda; K/\mathcal{O})) \). Then by definition (cf. (3.9)), we have

\[
(9.6) \quad \langle z \rangle = \lambda z_{n+2v} \langle z \rangle_{n, v, \lambda} \quad \text{for } z \in \overline{Z}_\alpha \text{ if } \lambda \text{ is a character of } Z_\alpha
\]

such that \( \lambda z_{n+2v} \) factors through \( \overline{Z}_\alpha \).

For each character \( \lambda: \overline{Z}_1 \to \mathcal{O}^\times \), we put

\[
\varphi^\text{ord}_q(v; U)[\lambda] = \{ x \in \varphi^\text{ord}_q(v; U) | x|\langle z \rangle = \lambda(z)x \text{ for all } z \in \overline{Z}_1 \}.
\]

**Theorem 9.4.** Let \( U \) be as above. Suppose that \( 0 \leq q \leq 1 \). Then, for each \( \beta > 0 \), the restriction map induces an isomorphism:

\[
e^{h} e\left( \bigoplus_{i=1}^{h} H^q(\overline{U}_\beta^1), 'L(n, v, \lambda; K/\mathcal{O}) \right) \equiv \varphi^\text{ord}_q(v; U)[\lambda z_{n+2v}]
\]

for each finite order character \( \lambda: \mathbb{Z}_1/\mathbb{Z}_\beta \to \mathbb{O}^\times \) such that \( \lambda z_{n+2v} \) factors through \( \overline{Z}_1 \) for each \( n \geq 0 \) with \( n \sim -2v \).

**Proof.** By Lemma 9.2, the assumption of Corollary 9.3 is satisfied for \( q = 1 \), and thus we have by (9.6) that

\[
eH^q(\overline{U}_\beta^1), 'L(n, v, \lambda; K/\mathcal{O}) \equiv \{ x \in eH^q(\overline{U}_\beta^1), 'L(n, v; K/\mathcal{O}) | x|\langle z \rangle = \lambda z_{n+2v} \text{ for all } z \in \overline{Z}_1 \}.
\]

On the other hand, the left-hand side of the above formula is isomorphic to \( \varphi^\text{ord}_q(v; U)[\lambda z_{n+2v}] \) under the restriction map. The case: \( q = 0 \) follows from Proposition 8.3 and (9.1).
Remark 9.5. The proof of Theorem 9.4 shows a little more general result for $0 \leq q \leq 1$ and for each $\beta \geq \alpha > 0$:

$$e\left(\bigoplus_{i=1}^h H^q\left(\Gamma_i(U^\alpha_{\beta}), ^tL(n, v, \lambda; K/\mathcal{O})\right)\right)$$

$$\cong \left\{ x \in \mathscr{V}_q^{\text{ord}}(v; U)|x|<\zeta = \lambda x_{n+2}\zeta(z)z \text{ for all } z \in \mathbb{Z}_\alpha \right\}.$$

10. Co-freeness of $\mathscr{V}_q^{\text{ord}}(v; N)$ over $\Lambda$

Let $U$ be either $U_1(N)$ or $U_1(N') \cap V_1(\ell')$ for an ideal $N$ outside $p$ and $\Sigma^B$ and for a prime ideal $\ell$ outside $Np$ and $\Sigma^B$. For each finite order character $\varepsilon: \mathbb{Z}_\alpha \to \mathbb{Q}_p^\times$, let $K(\varepsilon)$ denote the subfield of $\mathbb{Q}_p^\times$ generated by the values of $\varepsilon$ over $K$, and let $\mathcal{O}(\varepsilon)$ be the $p$-adic integer ring of $K(\varepsilon)$, and let $\beta(\varepsilon)$ denote the minimal integer such that $\text{Ker}(\varepsilon) \supseteq \mathbb{Z}_\beta$ and $\mathbb{Z}_\beta \cong \mathbb{Z}_\beta$. In this section, we always suppose that $B$ is a division algebra.

Theorem 10.1. Let $\alpha$ be a positive integer and suppose that $\mathbb{Q}_p^\times$ is torsion-free for all $i$. Let $q = 0$ or $1$. If $eH^q(X(U^\alpha_{\beta}), \mathcal{L}(n, v, \lambda; K(\lambda)/\mathcal{O}(\lambda)))$ is $p$-divisible for all finite order characters $\lambda: \mathbb{Z}_n \to \mathbb{Q}_p^\times$ such that $\lambda x_{n+2}$ factors through $\mathbb{Z}_\alpha$ and for all pair of integers $\gamma, \beta$ with $\alpha \leq \gamma \leq \beta$ and $\beta \geq \beta(\lambda)$, then the Pontryagin dual module $V_q^{\text{ord}}(v; U)$ of $\mathscr{V}_q^{\text{ord}}(v; U)$ is free of finite rank over the continuous group algebra $\mathcal{O}[[\mathbb{Z}_\alpha]]$ of $\mathbb{Z}_\alpha$.

Before proving the theorem, we prepare:

Lemma 10.2. Let $\mathcal{G}$ be a topological group isomorphic to a product of finitely many copies of $\mathbb{Z}_n$ and a finite group. Let $\mathcal{O}[[\mathcal{G}]]$ be the continuous group algebra of $\mathcal{G}$. For each finite order character $\varepsilon: \mathcal{G} \to \mathbb{Q}_p^\times$, let $P_\varepsilon: \mathcal{O}[[\mathcal{G}]] \to \mathbb{Q}_p^\times$ be the induced $\mathcal{O}$-algebra homomorphism. Then the subset of $\text{Spec}(\mathcal{O}[[\mathcal{G}]])$ consisting of the points $P_\varepsilon$ for all finite order characters $\varepsilon$ of $\mathcal{G}$ is Zariski dense.

Proof. What we have to show is that $\bigcap \text{Ker}(P_\varepsilon) = \{0\}$. Let $C(\mathcal{G}, \mathcal{O})$ denote the space of all continuous $\mathcal{O}$-valued functions on $\mathcal{G}$. Let $\text{Meas}(\mathcal{G}, \mathcal{O})$ be the $\mathcal{O}$-linear dual of $C(\mathcal{G}, \mathcal{O})$; i.e., $\text{Meas}(\mathcal{G}, \mathcal{O}) = \text{Hom}_e(C(\mathcal{G}, \mathcal{O}), \mathcal{O})$. We can identify $\text{Meas}(\mathcal{G}, \mathcal{O})$ with the space of bounded $p$-adic measures on $\mathcal{G}$ with values in $\mathcal{O}$. Let $\text{Meas}(\mathcal{G}, \Omega)$ be the space of all bounded $p$-adic measures on $\mathcal{G}$ with values in the $p$-adic completion $\Omega$ of $\mathbb{Q}_p$. $\text{Meas}(\mathcal{G}, \mathcal{O})$ is an $\mathcal{O}$-algebra under the convolution product and is isomorphic to $\mathcal{O}[[\mathcal{G}]]$ (e.g. [20]). Especially, for each finite order character $\varepsilon: \mathcal{G} \to \mathbb{Q}_p^\times$, the map: $\text{Meas}(\mathcal{G}, \mathcal{O}) \ni u \mapsto \int_{\mathcal{G}} \varepsilon du \in \Omega$ coincides with the algebra homomorphism $P_\varepsilon: \mathcal{O}[[\mathcal{G}]] \to \Omega$. Since the subspace of locally constant functions on $\mathcal{G}$ is dense in $C(\mathcal{G}, \mathcal{O})$, if $\int_{\mathcal{G}} \phi du = 0$ for all locally constant $\phi$, then $u = 0$. Note that every locally constant function $\phi$ can
be written as a linear combination of finite order characters \( \eta : \mathcal{O} \to \overline{\mathbb{Q}}_p^\times \) over \( \overline{\mathbb{Q}}_p \). Thus, if \( u \in \cap_i \ker(P_i) \), then \( \int_\eta \phi \, du = 0 \) for all locally constant \( \phi \) and hence \( u = 0 \). Q.E.D.

**Proof of Theorem 10.1.** By Theorem 6.3, without losing generality, we may replace \( K \) by its finite extension. Thus we may assume that we have a finite order character \( \eta : \mathbb{Z}_n / \mathbb{Z}_\gamma \to \mathcal{O} \) such that \( \epsilon \chi_{n+2v} \) factors through \( \mathbb{Z}_n \). We write \( V_{q, \text{ord}}(v; U) \) sometimes as \( V/\mathcal{O} \) to indicate its dependence on \( \mathcal{O} \). We have an exact sequence

\[
\begin{align*}
\varepsilon \left( X(U^\alpha), \mathcal{L}(n, v, \epsilon; K) \right) &\to eH^q(X(U^\alpha), \mathcal{L}(n, v, \epsilon; K/\mathcal{O})) \\
&\to eH^q(X(U^\alpha), \mathcal{L}(n, v, \epsilon; \mathcal{O})).
\end{align*}
\]

Since the middle term of (*) is \( p \)-divisible by assumption and the last term is of finite type as an \( \mathcal{O} \)-module (cf. [28, \S2]), \( \theta \) must be surjective. Hence the \( \mathcal{O} \)-rank of the Pontryagin dual of the middle term is finite. We write it as \( s \). Let \( \mathcal{O} \) act on \( \mathcal{O}^{\text{ord}}(v; U) \) via the action \( (z)_n \cdot v, \epsilon \) as in (9.6). We write \( A \) for \( \mathcal{O}[[\mathbb{Z}_n]] \). Then by Theorem 9.4, \( V \otimes_A A/P_\mathcal{O} A \) for the identity character \( \text{id} : \mathbb{Z} \to \mathbb{Q} \) is isomorphic to the Pontryagin dual module of the middle term of (*). Thus we know that \( V/P_\mathcal{O} V \cong \mathcal{O}^s \). Thus \( V \) is generated over \( A \) by \( s \)-elements, and we have a surjective morphism of \( A \)-modules \( \psi : A^s \to V \) which induces an isomorphism

\[
(A/mA)^s \cong V \otimes_A A/mA \quad \text{for the maximal ideal } m \text{ of } A.
\]

Note that \( A_\lambda = \mathcal{O}(\lambda)[[\mathbb{Z}_n]] \cong A \otimes \mathcal{O}(\lambda) \). We write \( K/\mathcal{O}(\lambda) \) for \( K(\lambda)/\mathcal{O}(\lambda) \). By using the same type of exact sequence as (*), we know

\[
eH^q\left( X(U^\beta), \mathcal{L}(n, v, \epsilon; K/\mathcal{O}(\lambda)) \right) = eH^q\left( X(U^\beta), \mathcal{L}(n, v, \epsilon; \mathcal{O}(\lambda)) \right) \otimes_\mathcal{O}(\lambda) (K/\mathcal{O}(\lambda))
\]

and by Theorem 6.3,

\[
eH^q\left( X(U^\beta), \mathcal{L}(n, v, \epsilon; K/\mathcal{O}(\lambda)) \right) \cong eH^q\left( X(U^\beta), \mathcal{L}(n, v, \epsilon; \mathcal{O}(\lambda)) \right) \otimes_\mathcal{O}(\lambda) \mathcal{O}(\lambda).
\]

Therefore, we see

\[
V/\mathcal{O} \otimes_\mathcal{O}(\lambda) \cong V/\mathcal{O} \otimes_A A_\lambda \cong V/\mathcal{O}(\lambda).
\]

Then for any finite order character \( \lambda : \mathbb{Z}_n \to \overline{\mathbb{Q}}_p^\times \), we know from the assumption that

\[
V/\mathcal{O} \otimes_A A_\lambda \cong \mathcal{O}(\lambda)^{s'} \quad \text{for some integer } s'.
\]

Then by (10.1a, b), \( s' \) must be equal to \( s \). Thus we have a surjection induced by \( \psi \):

\[
\mathcal{O}(\lambda)^{s} \cong (A_\lambda/P_\lambda A_\lambda)^s \to V/\mathcal{O} \otimes_A A_\lambda/P_\lambda A_\lambda \cong \mathcal{O}(\lambda)^{s}.
\]

From the exact sequence: \( 0 \to \ker(\psi) \to A^s \to V/\mathcal{O} \to \mathcal{O} \), we conclude in view
of the flatness of $\mathcal{O}(\lambda)/\mathcal{O}$ the exactness of
\[ 0 \to \text{Ker}(\psi) \otimes_{\mathcal{O}} \mathcal{O}(\lambda) \to A^\lambda \to V/\mathcal{O}(\lambda) \to 0. \]
Thus we know that $\text{Ker}(\psi) \otimes_{\mathcal{O}} \mathcal{O}(\lambda) \subset (P_{1,2,3,4})^{a} = P_{1}(A^\lambda)$. Since $\mathcal{O}(\lambda)$ is faithfully flat over $\mathcal{O}$, we conclude that $\text{Ker}(\psi) \subset (P_{1,2,3,4})^{a}$ for all finite order characters $\lambda$: $\mathbb{Z}_{p} \to \mathbb{Q}^{\times}$. Then by Lemma 10.2, we know the vanishing $\text{Ker}(\psi) = 0$, which proves the theorem.

**Corollary 10.3.** We suppose that $r = |I_B| = 0$ or 1. Then $V_{r,\text{ord}}(v; N)$ is a free module of finite rank over $\mathcal{O}[[\mathbb{Z}_{1}]]$, if one of the following two conditions is satisfied:

(10.2a) For a primitive $p$-th root of unity $\zeta_{p}$ in $\mathbb{C}$, $[F(\zeta_{0}): F] > 2$,

(10.2b) $\Gamma_{1}(Np)$ is torsion free for all $i$.

**Proof.** Firstly, we suppose that (10.2b) is satisfied. We write $U$ for $U_{1}(Np)$. When $r = 0$, the triviality of $\Gamma_{1}(Np)$ means that
\[ H^{0}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; K/\mathcal{O})) \cong L(n, v, \lambda; K/\mathcal{O}) \]
and thus it is $p$-divisible for all $0 < \alpha \leq \beta$ and $\lambda$. Hence we can apply Theorem 10.1 and get the result. When $r = 1$, it is known by [36, Propositions 8.1 and 8.2], if $\Gamma_{1}(Np)$ is without torsion, then
\[ H^{2}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; \mathcal{O})) \cong \bigoplus_{i} H_{0}(\Gamma_{1}(U_{\beta}^{a}), L(n, v, \lambda; \mathcal{O})), \]
and hence by Lemma 9.2, $eH^{2}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; \mathcal{O})) = 0$. From the exact sequence:
\[ eH^{1}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; K)) \to eH^{1}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; K/\mathcal{O})) \]
\[ \to eH^{2}(X(U_{\beta}^{a}), \mathcal{L}(n, v, \lambda; \mathcal{O})) = 0, \]
we know the $p$-divisibility of the middle term, and hence, the result follows from Theorem 10.1. Now we shall suppose (10.2a). On $F(\zeta_{p})$, the Frobenius element at each prime ideal $\mathfrak{p}$ of $v$ prime to $p$ acts by $\zeta_{p} \to \zeta_{p}^{\mathcal{N}(\mathfrak{p})}$ for the norm map $\mathcal{N}: F \to \mathbb{Q}$. Since $[F(\zeta_{p}): F] > 2$, we can find a prime $\mathfrak{p}$ of $N$ unramified over $\mathbb{Z}$ such that $\mathcal{N}(\mathfrak{p}) \neq \pm 1$ mod $p$. Since the degree $d$ of $X(V)/X(U)$ for $V = U_{1}(Np) \cap V_{1}(\mathfrak{p})$ divides $(\mathcal{N}(\mathfrak{p}) + 1)(\mathcal{N}(\mathfrak{p}) - 1)$, $d$ is prime to $p$. By Lemma 7.1, $\Gamma_{1}(V)$ is torsion-free. The similar argument as in the case where (10.2b) is satisfied shows that $V_{r,\text{ord}}(v; V)$ is free of finite rank over $\mathcal{O}[[\mathbb{Z}_{1}(N\mathfrak{p})]]$. Since $\mathbb{Z}_{1} = \mathbb{Z}_{1}(N\mathfrak{p})$ can be identified with a subgroup of $\mathbb{Z}_{1}(N\mathfrak{p})$ (because the covering degree of $\mathbb{Z}_{1}(N\mathfrak{p})/\mathbb{Z}_{1}$ is prime to $p$), $V_{r,\text{ord}}(v; V)$ is free over $\mathcal{O}[[\mathbb{Z}_{1}]]$. For the trace map $\text{Tr}_{U/V} = \text{Tr}: \mathcal{Y}_{r,\text{ord}}(v; V) \to \mathcal{Y}_{r,\text{ord}}(v; N)$ and the restriction map $\text{res}_{U/V} = \text{res}: \mathcal{Y}_{r,\text{ord}}(v; V) \to \mathcal{Y}_{r,\text{ord}}(v; V)$, we know that $\text{Tr} \circ \text{res}$ is the multiplication by $d$. Thus $V_{r,\text{ord}}(v; N)$ is a direct summand of the $\mathcal{O}[[\mathbb{Z}_{1}]]$-free module $\mathcal{Y}_{r,\text{ord}}(v; V)$ and is hence free over $\mathcal{O}[[\mathbb{Z}_{1}]]$. 


As for the structure of the $\Lambda$-module, we have:

**Corollary 10.4.** Suppose that $r = |I_B| = 0$ or 1. Let $P$ be the product of all distinct prime factors of $p$ in $t$. Suppose that $p > 2$ and further assume either (10.2a) or the following condition:

(10.2c) $\overline{\Gamma}_i(NP)$ is torsion free for all $i$.

Then $V^\ord_r(v; N)$ is $\Lambda$-free of finite rank, where $\Lambda = \emptyset[[W]]$ is the continuous group algebra of the torsion-free part $W$ of $\overline{Z}(N)$.

**Proof.** We shall prove the assertion only in the case $r = 1$, since the other case can be handled more easily. So far, we have worked with the filtration $Z_a \subset Z(N)$, but we can define another filtration of subgroups $Z^a$ of $Z(N)$ given by the kernel of the natural map: $Z \rightarrow Cl_F(NP^a)$. Since $p$ is odd, even if one defines $\overline{Z}^a$ by the kernel of \("$\overline{Z} \rightarrow \overline{Cl}_F(NP^a)$",\) we can identify $\overline{Z}^a$ with $Z^a$ naturally. Hereafter we always identify these two groups. These two filtrations are cofinal; i.e., $Z^a \supseteq Z_a$ and $Z_a \supseteq Z^{ma}$ if $p$ divides $P^m$. If $p$ is unramified in $t$, then the two filtrations coincide. We can define $\Delta^a_{C_F}$ similarly to $\Delta^a_{F}$ by replacing $p$ by $P$ in the definition of $\Delta^a_{F}$ in Section 7 and define $U_{(\beta)}^{(a)}$ by $U \cap \Delta^a_{C_F}$ for $U$ as in Theorem 9.4. By scrutinizing every step ascending towards the proof of Theorem 9.4, one can check that the corresponding statement to Theorem 9.4 for $U_{(\beta)}^{(1)}$ is true. Put

$$\varphi(i) = \lim_{\alpha} eH_1^{\ell}(\overline{\Gamma}_i(NP^a), \ell L(n, v; K/\emptyset))$$

by choosing $n \geq 0$ with $n \sim -2v$. Then in exactly the same manner as in the proof of Theorem 10.1 and Corollary 10.3, we can prove that $V_r^{\ord}(v; N) = \emptyset V(i)$ is $\emptyset[[Z]]$-free, where $V(i)$ is the Pontryagin dual of $\varphi(i)$. Put $W^1 = W \cap Z^1$. Since $[Z^0 : Z^1]$ is prime to $p$, the natural projection: $W \rightarrow Cl_F(1)$ induces an injection: $W/W^1 \rightarrow Cl_F(1)$. We can identify the set of connected components of $X(U)$ for $U = U_{(\beta)}(NP^a)$ (for any $a$) with the group $Cl_F(1)$ via the correspondence: $G_{\emptyset t} \rightarrow Cl_{F^\infty} \rightarrow F^\times \nu(t) \tilde{\delta}_F^\times$ in $Cl_F(1)$. Thus we can write $\varphi_r^{\ord}(v; N) = \emptyset_{\ell \in Cl_F(1)} \varphi(\ell)$. Then the action of $w \in W/W^1 \subset Cl_F(1)$ interchanges the connected components according to the multiplication of $\nu(w) = w^2$ in $Cl_F(1)$.

If $p > 2$, the map: $w \mapsto w^2$ gives an automorphism of the image $\overline{W}$ of $W$ in $Cl_F(1)$. By taking a coset decomposition $Cl_F(1) = \bigcup_{j} \overline{W} \cdot j$, put $V_0 = \emptyset V(j)$.

Then $V_0$ is $\emptyset[[W]]$-free, and $V_r^{\ord}(v; N) = \text{Ind}_W^F(V_0) \cong V_0 \otimes_{\emptyset[[W]]} \Lambda$ is $\Lambda$-free.

11. Proof of Theorems 3.2 and 3.3

When we consider the field $F \neq \mathbb{Q}$, we always fix here and in the following section a quaternion algebra $B$ over $F$ unramified at all finite places and
\[ r = |I_B| \leq 1. \] Therefore \( \B \) is a division quaternion algebra and \( X(U) \) is always compact. Such a quaternion algebra always exists if \( [F : \mathbb{Q}] > 1 \), and we have a relation: \( [F : \mathbb{Q}] \equiv r \mod 2 \). When \( F = \mathbb{Q} \), we take \( M_2(\mathbb{Q}) \) as \( B \).

**Proof of Theorem 3.2.** Firstly we suppose that \( F \neq \mathbb{Q} \). Let \( \mathcal{O} \) be a finite extension of \( \mathbb{Q}_p \) containing \( K_0 \) as in Section 1. Let \( \mathcal{O} \) be the \( p \)-adic integer ring of \( K \). It is sufficient to prove the result over \( \mathcal{O} \) since \( \mathcal{A}_{k, \mathcal{O}}(N_{p^\infty}; \mathcal{O}) \cong \mathcal{A}_{k, \mathcal{O}}(N_{p^\infty}; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} \mathcal{O} \) for the \( p \)-adic closure \( \mathcal{O}(v) \) of \( \mathcal{O}(v) \) in \( \mathbb{Q}_p \). We consider

\[
\gamma_r'(0; N) = \lim_\to H'(X_{\alpha}, K/\mathcal{O}) \text{ for } X_{\alpha} = X(U_1(N_{p^\alpha})).
\]

We have an exact sequence: \( H^1(X_{\alpha}, K) \to H^1(X_{\alpha}, K/\mathcal{O}) \to H^2(X_{\alpha}, \mathcal{O}) \). When \( r = 1 \), \( H^2(X_{\alpha}, \mathcal{O}) \cong 0 \); thus, \( H^1(X_{\alpha}, K/\mathcal{O}) \) is \( p \)-divisible. Writing \( \gamma'_a \) for \( H^1(X_{\alpha}, K/\mathcal{O}) \), we can identify \( \mathcal{A}_{2t, t}(N_{p^\alpha}; \mathcal{O}) \) with the subalgebra of \( \text{End}_{\mathcal{O}}(\gamma'_a) \) generated over \( \mathcal{O} \) by \( T(n) \) for all \( n \) by Theorem 6.2 (in this case: \( v = 0 \), \( T_0(n) = T(n) \) for all \( n \)). Now we consider the case: \( r = 0 \). In this case, let \( S_\alpha \) (resp. \( T \)) be the space of functions \( f: \mathcal{G}_\alpha^B \to K/\mathcal{O} \) (resp. \( f: \mathcal{G}_x^x \to K/\mathcal{O} \)), such that \( f(axu) = f(a) \) for \( a \in \mathcal{G}_\alpha^B \) and \( u \in U_1(N_{p^\alpha})G_{\alpha}^{\infty} \) (resp. \( a \in \mathcal{G}_x^x \) and \( u \in S_x^{\infty} \)). Since the reduced norm map \( \nu: \mathcal{G}_\alpha^B \to \mathcal{G}_x^x \) induces an injection \( \nu^*: T \to S_\alpha \), the quotient \( \gamma_a = S_\alpha/\nu^*(T) \) is \( p \)-divisible, and therefore, we can identify, by Theorem 6.2, \( \mathcal{A}_{2t, t}(N_{p^\alpha}; \mathcal{O}) \) with the subalgebra of \( \text{End}_{\mathcal{O}}(\gamma'_a) \) generated by \( T(n) \). For each \( a \in F_+^x = F^x \cap F_{\infty}^x \), we consider the quadratic extension \( F(\sqrt{-a}) \), which is totally imaginary. Since \( B \) is unramified at all finite places of \( F \), \( F(\sqrt{-a}) \) can be embedded into \( B \) ([36, (9.2.6)]), and hence, \( a = v(\sqrt{-a} ) \in v(B^x) \). That is, \( v: \mathcal{G}_x^x \to F^x_{\chi} \) is surjective. For \( a \in F_+^x \), \( y \in F_{A+}^x = F_+^x \cap F_{\infty}^x \), and \( c \in S_{\alpha}^\infty \), we choose \( b \in \mathcal{G}_x^x \), \( x \in \mathcal{G}_x^x \) and \( u \in U_1(N_{p^\alpha})G_{\alpha}^{\infty} \) such that \( v(b) = a \), \( v(x) = y \) and \( v(u) = c \). If \( f \in S_\alpha \) is of the form \( \nu^*(\phi) \) for \( \phi: F_{A+}^x \to K/\mathcal{O} \), then \( \phi(abc) = \nu(\phi(abc)) = f(abc) = f(x) = \phi(y) \) and thus \( f \in \nu^*(T) \). That is if \( f \in \gamma_a \) becomes trivial in \( \gamma_a \) for \( \alpha \leq \beta > \alpha > 0 \), then \( f \) must be zero in \( \gamma_a \). Therefore the natural map: \( \gamma_a \to \gamma'_a \) is injective. When \( r = 1 \), by virtue of a result of Shimura as in Theorem 8.7, the natural map: \( \gamma_a \to \gamma'_a \) \((\alpha < \beta \leq \infty)\) has a finite kernel. Then in either case: \( r = 0 \) or \( 1 \), the natural \( \mathcal{O} \)-algebra homomorphism \( \rho^0_{\beta}: \mathcal{A}_{2t, t}(N_{p^\alpha}; \mathcal{O}) \to \mathcal{A}_{2t, t}(N_{p^\alpha}; \mathcal{O}) \) \((\alpha < \beta \leq \infty)\) can be defined by the commutative diagram:

\[
\begin{array}{ccc}
\gamma_a & \xrightarrow{\text{res}} & \gamma_a' \\
\downarrow \rho^0_{\beta}(h) & & \downarrow h \\
\gamma_a & \xrightarrow{\text{res}} & \gamma_a'
\end{array}
\]
where we understand that $\mathcal{Y}_\infty = \lim_{\alpha} \mathcal{Y}_\alpha$. Especially $\mathcal{H}_{2t_0}(Np^\infty; \mathcal{O})$ acts faithfully on $\mathcal{Y}_\infty$ and hence is naturally embedded into $\text{End}_A(\mathcal{Y}_\infty)$. Now we consider the morphism defined in (8.10) for $n \sim 0 (n > 0)$:

$$\iota: \bigoplus_i H^i_!(\Gamma_i^!(Np^n), iL(n, 0; K/\mathcal{O})) \to \mathcal{Y}_r(0; N).$$

When $r = 0$ and $n \sim 0$, the space on the left-hand side can be identified with the space $S_{k, k-t}(Np^n; K/\mathcal{O}) (k = n + 2t)$ of functions $f: G^n_k \to L(n, 0; K/\mathcal{O})$ such that

$$f(axu) = f(x) \cdot a^n$$

for $a \in G^n_k$ and $u \in U_i(Np^n)G_{\infty+}$. Since for $u \in U_i(Np^n)G_{\infty+}$ with $u_p = \left(\begin{smallmatrix} a & \cdot \\ 0 & b \end{smallmatrix}\right)$,

$$f_n(xu) = f_n(x)a^n,$$

we know that if $f \in \nu^*(T)$, then the value of $f_n$ is annihilated by $a^n - 1$ for all $a \in \mathfrak{p}_p'$ with $a \equiv 1 \mod p^{2t}$. Thus we can choose $\beta > 0$ independently of $f$ so that $p^\beta f \in \text{Ker}(\iota)$ if $f \in \nu^*(T)$. Then by Theorem 8.7, the map induced by $\iota$:

$$I_0: \bigoplus_i H^i_!(\Gamma_i^!(Np^n), iL(n, 0; K/\mathcal{O})) \to \mathcal{Y}_\infty$$

is of finite kernel. This fact is also true by Theorem 8.7 in the case: $r = 1$ since $[F : Q] = 1 \mod 2$ in this case. Let $\mathcal{Y}_{\alpha}^n$ be the image of $\bigoplus_i H^i_!(\Gamma_i^!(Np^n), iL(n, 0; K))$ in $\bigoplus_i H^i_!(\Gamma_i^!(Np^n), iL(n, 0; K/\mathcal{O}))$. Then $\mathcal{Y}_{\alpha}^n$ is the $p$-divisible part of the latter module. Then, by Theorem 6.2, for each $n \sim 0$ ($n \geq 0$), $\mathcal{H}_{k, k-t}(Np^n; \mathcal{O}) (k = n + 2t)$ can be identified with the $\mathcal{O}$-subalgebra of $\text{End}_G(\mathcal{Y}_{\alpha}^n)$ generated by $T(n)$ for all $n$. Since $I_0: \mathcal{Y}_{\alpha}^n \to I_0(\mathcal{Y}_{\alpha}^n) \subset \mathcal{Y}_\infty$ is an isogeny, $\mathcal{H}_{k, k-t}(Np^n; \mathcal{O})$ can be also identified with the $\mathcal{O}$-subalgebra of $\text{End}_G(I_0(\mathcal{Y}_{\alpha}^n))$ generated by $T(n)$ for all $n$. The restriction of operators in $\mathcal{H}_{2t, t}(Np^\infty; \mathcal{O}) \subset \text{End}_G(\mathcal{Y}_\infty)$ to the subspace $I_0(\mathcal{Y}_{\alpha}^n)$ induces a surjective $\mathcal{O}$-algebra homomorphism $I_0^*: \mathcal{H}_{2t, t}(Np^\infty; \mathcal{O}) \to \mathcal{H}_{k, k-t}(Np^n; \mathcal{O})$ which takes $T(n)$ to $T(n)$ because of Theorem 8.1. Thus we have a commutative diagram for $0 < \alpha < \beta < \infty$:

$$\begin{array}{ccc}
\mathcal{H}_{2t, t}(Np^\infty; \mathcal{O}) & \xrightarrow{I_0^*} & \mathcal{H}_{k, k-t}(Np^n; \mathcal{O}) \\
\downarrow{I_0^*} & & \downarrow{\phi^0} \\
\mathcal{H}_{k, k-t}(Np^\beta; \mathcal{O}) & & \end{array}$$

where $\phi^0$ is the natural projection map given in Section 3. These maps are
surjective morphisms in the category of compact rings where the projective limit of surjective morphisms is always surjective. By taking the projective limit of $I^*_\alpha$, we have a surjective \mathcal{A}lgebra homomorphism

$$I^*_\alpha: \mathfrak{h}_{2t, t}(Np^\infty; \emptyset) \to \mathfrak{h}_{k, k-t}(Np^\infty; \emptyset) \quad \text{for } k \geq 2t \text{ and } k \sim 0,$$

which takes $T(n)$ to $T(n)$. When $F = \mathbb{Q}$, we have to replace $H^1$ by the parabolic cohomology groups $H^1_p$ defined in [36, Chap. 8] (see also [14, \S 4]). Then every step in the above argument for $F \neq \mathbb{Q}$ can be checked for $H^1_p$ and we obtain the result even for $F = \mathbb{Q}$.

Now we shall prove the injectivity of $I^*_\alpha$ when $f \neq \mathbb{Q}$. Put $\mathcal{Y}^\alpha_n = \varprojlim_a \mathcal{Y}^\alpha_n$. We consider the map $I^*_\alpha = \lim_a I^*_\alpha: \mathcal{Y}^\alpha_n \to \mathcal{Y}^\alpha_\infty$. By definition, we have a commutative diagram: for all $h \in \mathfrak{h}_{2t, t}(Np^\infty; \emptyset)$,

$$
\begin{array}{ccc}
\mathcal{Y}^\alpha_n & \xrightarrow{I^*_\alpha} & \mathcal{Y}^\alpha_\infty \\
\downarrow I^*_\alpha(h) & & \downarrow h \\
\mathcal{Y}^\alpha_n & \xrightarrow{\mathcal{Y}^\alpha_\infty} & \mathcal{Y}^\alpha_\infty
\end{array}
$$

Thus if $I^*_\alpha$ is surjective, $I^*_\alpha$ must be injective since $\mathfrak{h}_{2t, t}(Np^\infty; \emptyset) \subset \text{End}_A(\mathcal{Y}^\alpha_\infty)$. If $r = 0$, by identifying $\mathcal{Y}^\alpha_\infty$ with $S_\infty/p^\infty(T)$ and $\mathcal{Y}^\alpha_n$ with the $p$-divisible subspace of $\varprojlim_a S_{k, k-t, \phi}(Np^a; K/\emptyset)$, we see easily the surjectivity of $I^*_\alpha$, and the result follows. Thus we shall prove the surjectivity of $I^*_\alpha$ in the case of $r = 1$. Put $\mathcal{Y}^\alpha_n = \bigoplus_i H^1_i(\mathfrak{h}_{i}(Np^a), \mathcal{L}(n, 0; K/\emptyset))$. Then we have an exact sequence for $n > 0$:

$$0 \to \mathcal{Y}^\alpha_n \to \mathcal{Y}^\alpha_n \to H^2(\mathfrak{h}_{i}(Np^a), \mathcal{L}(n, 0; K/\emptyset)) \to 0.$$

Then, by [36, Prop. 8.1 and 8.2], there is an isogeny

$$f_\alpha: H^2(\mathfrak{h}_{i}(Np^a), \mathcal{L}(n, 0; K/\emptyset)) \to \mathcal{L}(n, 0; K/\emptyset)/D^i_\alpha,$$

$$g_\alpha: \mathcal{L}(n, 0; K/\emptyset)/D^i_\alpha \to H^2(\mathfrak{h}_{i}(Np^a), \mathcal{L}(0, n; K/\emptyset))$$

such that $f_\alpha \circ g_\alpha = g_\alpha \circ f_\alpha = M \cdot \text{id}$, where $M$ is the least common multiple of the order of all torsion elements in $\mathfrak{h}_{i}(Np^a)$ and

$$D^i_\alpha = \sum_{\gamma \in \Gamma_{i}(Np^a)} (\gamma - 1) \cdot \mathcal{L}(n, 0; K/\emptyset).$$

Thus by taking the injective limit relative to $\alpha$, we have an exact sequence: $0 \to \mathcal{Y}^\alpha_\infty \to \mathcal{Y}^\alpha_n \to \mathcal{X} \to 0$, where $\mathcal{X} = \varprojlim_a (\bigoplus_i H^2(\mathfrak{h}_{i}(Np^a), \mathcal{L}(n, 0; K/\emptyset)))$. We shall define a map

$$\phi^\alpha_\beta: \mathcal{L}(n, 0; K/\emptyset)/D^i_\alpha \to \mathcal{L}(n, 0; K/\emptyset)/D^i_\beta \quad \text{for } \beta > \alpha.$$
by the transfer map given by the correspondence: \( x \mapsto \sum \gamma^{-1}x \) for a decomposition \( \Gamma'(Np^\alpha) = \bigcup \Gamma(Np^\beta) \gamma \). Then the injective limit for \( \mathcal{X}' \) is compatible with the injective limit \( \mathcal{X}' = \lim_{\alpha} \left( \bigcup (\bigoplus L(n, 0; \mathcal{O})/D^i_{\alpha}) \right) \) relative to \( \phi_{\alpha}^p \) under \( f_\alpha \) and \( g_\alpha \). (This follows from the proof of the existence of \( f_\alpha \) and \( g_\alpha \) given in [36].) Thus we have \( f: \mathcal{X} \rightarrow \mathcal{X}' \) and \( g: \mathcal{X}' \rightarrow \mathcal{X} \) such that \( f \circ g = g \circ f = M \cdot \text{id} \) for a suitable positive integer \( M \). Let \( \Gamma^i_\alpha \) be the closure of \( \Gamma(Np^\alpha) \) in \( G_p = GL_2(F_p) \). Note that \( D^i_\alpha \) is a closed submodule of \( 1L(n, 0; \mathcal{O}) \). Therefore for any \( \gamma \in \Gamma^i_\alpha \), we know that \( (\gamma - 1)1L(n, 0; \mathcal{O}) \subset D^i_\alpha \). Especially, for every \( a \in \mathfrak{i}_p^\infty \) with \( a = 1 \mod p^\alpha \), put \( \gamma_a = \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right) \in GL_2(F_p) \); then, \( \gamma_a \in \Gamma^i_\alpha \). Thus \( D^i_\alpha \supset D_\alpha = \sum \gamma_a \mathcal{O} \). Note that \( \gamma_a \) acts on the monomial \( X^n \) as \( \gamma_a X^n = a^n X^n \). Therefore, on the image of \( X^n \) in \( 1L(n, 0; \mathcal{O})/D_\alpha \), every element \( \gamma \) of \( \Gamma(Np^\alpha) \) acts trivially since \( \gamma \equiv (1, 0) \mod p^\alpha \mathcal{O} \). Therefore we know that \( i(\phi^p_{\alpha}(x)) = [\Gamma(Np^\alpha): \Gamma(Np^\beta)] \gamma(x) \). Since \( [\Gamma(Np^\alpha): \Gamma(Np^\beta)] \) is a \( p \)-power whose exponent increases accordingly to \( \beta \), \( I_\infty \) is the zero map on \( \mathcal{X}' \). Therefore, we have the commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \mathcal{Y}_n^\infty \longrightarrow \mathcal{Y}_n^\infty \longrightarrow \mathcal{X} \longrightarrow 0 \\
\downarrow I_x \quad \downarrow I_x \quad \downarrow I_x \\
0 \longrightarrow I_\infty(\mathcal{Y}_n^\infty) \longrightarrow \mathcal{Y}_\infty \longrightarrow \mathcal{Y} \longrightarrow 0
\end{array}
\]

and \( \mathcal{Y} \) is annihilated by \( M \). Since \( \mathcal{Y} \) is a surjective image of the \( p \)-divisible group \( \mathcal{Y}_\infty \), it must be trivial. This shows the surjectivity of \( I_x: \mathcal{Y}_n^\infty \rightarrow \mathcal{Y}_\infty \), which finishes the proof when \( F \neq \mathbb{Q} \). When \( F = \mathbb{Q} \), as seen in [14, Lemma 7.2], we already have a surjective inverse map of \( I^*: \mathcal{H}_{k, k}(Np^\infty; \mathcal{O}) \rightarrow \mathcal{H}_{2r, k}(Np^\infty; \mathcal{O}) \), and hence \( I_x^* \) must be an isomorphism (in [14], we made the assumption: \( p \geq 5 \), but this condition is not necessary for the proof of [14, Lemma 7.2]).

**Proof of Theorem 3.3.** It is sufficient to prove the assertion over \( \mathcal{O} \) since \( \mathcal{H}_{k, w}(Np^\infty; \mathcal{O}) \cong \mathcal{H}_{k, w}(Np^\infty; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} \mathcal{O} \). Fix \( v \) as in Section 3, and define \( k = n + 2t \), \( w = v + k - t \) for each \( n \geq 0 \) with \( n \sim 2v \). We firstly suppose that \( F \neq \mathbb{Q} \). Let \( \mathcal{Y}_{n, v} = \mathcal{Y}_{n, v}(N) \) be the image in \( e(\bigoplus L(Np^\alpha), 1L(n, v; K/\mathcal{O})) \) of \( e(\bigoplus L(Np^\alpha), 1L(n, v; K)) \). By definition, the restriction map takes \( \mathcal{Y}_{n, v} \) into \( \mathcal{Y}_{n, v} \) for \( \beta > \alpha \). Put \( \mathcal{Y}_{n, v}(N) = \lim_{\alpha} \mathcal{Y}_{n, v}(N) \). We know that \( \mathcal{Y}_{n, v} \subset \mathcal{Y}_{\infty, v}(v; N) \). We choose another ideal \( M \) prime to \( p \) such that \( N \mid M \). \( \Gamma^i(M) \) is torsion-free for all \( i \). Such an ideal \( M \) exists by Lemma 7.1. Then \( \mathcal{Y}_{n, v}(M) \) coincides with \( \mathcal{Y}_{\infty, v}(v; M) \). Note that the trace map of \( \bigoplus H^1(\Gamma^i, 1L(n, v; K)) \) is always surjective
by Theorems 6.2 and 6.3 and is compatible with the injective systems \( \{ \mathcal{X}_a, n(N) \}_a \) and \( \{ \mathcal{X}_a, n(M) \}_a \). Thus \( \mathcal{X}_\infty, n(N) \) is the surjective image of \( \mathcal{X}_\infty, n(M) = \mathcal{X}_r^{\text{ord}}(v; M) \) and is independent of \( n \) if \( n \equiv -2v \) because the trace map is independent of weight \( n \) by Theorem 8.1. Since \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \) can be identified with the subalgebra of \( \text{End}_A(\mathcal{X}_\infty, n(M)) \) and the restriction map \( \mathcal{X}_a, n(N) \to \mathcal{X}_\infty, n(N) \) is injective by Remark 9.5, \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \) can be identified with the subalgebra of \( \text{End}_A(\mathcal{X}_\infty, n(M)) \) generated topologically over \( \Lambda \) by \( T_0(n) \) for all \( n \in \mathbb{I}(1) \). Since the Pontryagin dual module \( \mathcal{V}_r^{\text{ord}}(v; N) \) is a \( \Lambda \)-module of finite type by Theorem 9.4, the Pontryagin dual module \( \mathcal{V}_\infty, n(N) \) of \( \mathcal{X}_\infty, n(N) \) is also of finite type, and hence \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \) is a \( \Lambda \)-module of finite type since \( \Lambda \) is noetherian. Since \( \mathcal{V}_r^{\text{ord}}(v; M) \) is \( \mathcal{O}[[W]] \)-free by Corollary 10.3 and \( \Lambda \) is finite and faithfully flat over \( \mathcal{O}[[W]] \), \( \mathcal{V}_r^{\text{ord}}(v; M) \) is \( \Lambda \)-torsion-free. Since \( \mathcal{V}_\infty, n(N) \) can be identified with a \( \Lambda \)-submodule of \( \mathcal{V}_r^{\text{ord}}(v; M) \), \( \mathcal{V}_\infty, n(N) \) is \( \Lambda \)-torsion-free, and hence \( \text{End}_A(V_\infty, n(N)) \) is \( \Lambda \)-torsion-free (even, in fact, \( \Lambda \)-reflexive). Therefore \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \), which is a \( \Lambda \)-submodule of \( \text{End}_A(V_\infty, n(N)) \), is \( \Lambda \)-torsion-free. Since \( \mathcal{V}_\infty, n(N) \) is independent of \( n \) and \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \) is generated over \( \Lambda \) by \( T_0(n) \) for all \( n \), \( k_{k, u}^{\text{ord}}(Np^\infty; \mathcal{O}) \) is independent of \( n \) and only depends on \( v \) mod \( \mathbb{Z} \cdot t \). This shows the assertion when \( F \neq \mathbb{Q} \). The case: \( F = \mathbb{Q} \) can be handled in exactly the same manner as above, replacing the usual cohomology groups \( H^1 \) by the parabolic ones \( H^1_p \). The case: \( F = \mathbb{Q} \) under an additional assumption: \( p \geq 5 \) has already been treated[14, §1]. However, by carefully analyzing the proof of [14, Th. 3.1], one finds that the assumption: \( p \geq 5 \) can be removed if \( \mathcal{V}_0(N) \) is torsion-free. Then the above argument in the case where \( F \neq \mathbb{Q} \) works well even for \( F = \mathbb{Q} \).

12. Proof of Theorems 3.4 and 3.6 and Corollary 3.7

When \( r = |I_B| = 1 \), \( C_{\infty}/C_{\infty^+} \) is an abelian group of order 2. By the result in Section 7, we can let \( C_{\infty}/C_{\infty^+} \) act on \( \mathcal{V}_r^{\text{ord}}(v; N) \). We write simply \( \mathcal{V} \) for \( \mathcal{V}_r^{\text{ord}}(v; N) \) and put \( \mathcal{V}_\pm = \{ x \pm (x|c)|x \in \mathcal{V} \} \) for the generator \( c \) of \( C_{\infty}/C_{\infty^+} \). If \( \mathcal{V} \) is \( p \)-divisible, then \( \mathcal{V}_\pm \) is also \( p \)-divisible and \( \mathcal{V} = \mathcal{V}_+ + \mathcal{V}_- \). When \( p > 2 \), \( \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_- \). Before proving Theorem 3.4, we shall show:

**Theorem 12.1.** Let \( K \) be a finite extension of \( \mathbb{Q}_p \) containing \( K_0 \) as in Section 1 and \( \mathcal{O} \) be its \( p \)-adic integer ring. When \( r = 1 \), let \( V \) denote the Pontryagin dual module of one of \( \mathcal{V}_\pm \) and when \( r = 0 \), let \( V \) denote the Pontryagin dual module of \( \mathcal{V}_0^{\text{ord}}(v; N) \). Then we have an isomorphism of Hecke modules:

\[
V_p = V \otimes_\Lambda \Lambda_p \cong h_v^{\text{ord}}(N; \mathcal{O}) \otimes_\Lambda \Lambda_p \quad \text{for all } P \in \mathcal{F}_{\text{alg}}(\Lambda) \text{ with } n(P) \geq 2v,
\]

where \( \Lambda_p \) is the localization of \( \Lambda \) at \( P \). Especially \( h_v^{\text{ord}}(N; \mathcal{O}) \otimes_\Lambda \Lambda_p \) is free of finite rank over \( \Lambda_p \) for all \( P \in \mathcal{F}_{\text{alg}}(\Lambda) \) with \( n(P) \geq 2v \).
Proof. For simplicity, we write \( h_p \) for \( \mathcal{H}_v^\text{ord}(N; \emptyset) \otimes \Lambda \Lambda_p \). Firstly suppose that \( \widetilde{\Gamma}_i^i(N P) \) is torsion free for all \( i \) for \( \mathcal{P} \) as in Corollary 10.4. We also suppose that \( p > 2 \) or \( r = 0 \). Then \( V \) is free of finite rank over \( \Lambda \) by Corollary 10.4. Let \( \Lambda' = \emptyset[[W]] \) and put \( Q = P \cap \Lambda' \). Then \( Q \) corresponds to the restriction of the character \( \chi_{n(P)} \mathcal{E}_P \) to \( W_1 \). Let \( \Lambda'_0 \) be the localization of \( \Lambda' \) at \( Q \) and put \( V_Q = V \otimes \Lambda', \Lambda'_Q \). By Theorem 9.4, we know from (7.6) that

\[
V_Q/QV_Q \cong \bigoplus \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon; K)
\]

for \( k = n(P) - 2v + 2t \) and \( w = v + k - t \),

where \( \varepsilon \) runs over all characters of \( W \) whose restriction to \( W_1 \) coincides with \( \varepsilon_p \). This shows that \( \mathcal{H}_p/\mathcal{P} \mathcal{H}_p \cong \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon_p; K) \). By definition, we have a natural surjection: \( h_p/P h_p \rightarrow \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon_p; K) \). Let \( \tilde{x} \) be the element in \( \mathcal{H}_p/\mathcal{P} \mathcal{H}_p \) corresponding to the identity of \( \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon_p; K) \). Take \( x \in \mathcal{H}_p \) such that \( x \mod P = \tilde{x} \), and define \( \phi: h_p \rightarrow \mathcal{P} \mathcal{H}_p \) by \( \phi(h) = hx \). Then, by Nakayama’s lemma, \( \phi \) is surjective. It is clear from the proof of Theorem 3.2 and 3.3 that \( h_p \) acts faithfully on \( V_p \) and hence \( \phi \) is an isomorphism. For general \( N \), we take an ideal \( M \) prime to \( p \) such that \( \widetilde{\Gamma}_i^i(M P) \) is torsion-free for all \( i \). Let us consider \( Tr: \mathcal{V}_r^\text{ord}(v; MN) \rightarrow \mathcal{V}_r^\text{ord}(v; N) \) and res: \( \mathcal{V}_r^\text{ord}(v; N) \rightarrow \mathcal{V}_r^\text{ord}(v; MN) \). Then \( Tr \circ \text{res} \) coincides with the multiplication of a positive integer \( d \). Therefore, if \( p > 2 \) or \( r = 0 \), we conclude from the result for level \( MN \) that \( V_p \) is \( \Lambda_p \)-free and

\[
\mathcal{H}_p/\mathcal{P} \mathcal{H}_p \cong \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon_p; K).
\]

Then the same argument as above shows the assertion. When \( p = 2 \) and \( r = 1 \), the kernel of the natural map: \( \mathcal{V}_r^{+} \otimes \mathcal{V}_r^{-} \rightarrow \mathcal{V}_r^{+} \) is annihilated by 2, and thus for the Pontryagin dual modules \( V^+, V^- \) and \( V^* \) of \( \mathcal{V}_r^{+} \), \( \mathcal{V}_r^{-} \) and \( \mathcal{V}_r^{+} \), respectively,

\[
(V^+ \otimes \Lambda \Lambda_p) \oplus (V^- \otimes \Lambda \Lambda_p) \cong V^* \otimes \Lambda \Lambda_p.
\]

This combined with the above argument shows the result.

Proof of Theorem 3.4. By Theorem 12.1, if \( K \) contains \( K_0 \) as in (1.1), we know from Theorem 9.4 and (7.6) that

\[
h_v^{\text{ord}}(N; \emptyset) \otimes \Lambda \Lambda_p/P \Lambda_p \cong \mathcal{H}_p/\mathcal{P} \mathcal{H}_p \cong \mathcal{H}_k^{\text{ord}}(N_{p}^{a(P)}, \varepsilon_p; K)
\]

as in the theorem. By Theorems 3.1 and 3.3, we know that \( \mathcal{H}_v^{\text{ord}}(N; \emptyset) \cong \mathcal{H}_v^{\text{ord}}(N; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} \emptyset \). Then the general case follows from the result for \( \emptyset \).

Before proving Theorem 3.6 and Corollary 3.7, we prepare some lemmas.

**Lemma 12.2.** Let \( p \) be a prime factor of \( p_n \) in \( n \) and \( N \) be an ideal prime to \( p \). Let \( \lambda: \mathcal{H}_k^{\text{ord}}(N_{p}^{a}; \mathcal{Q}) \rightarrow \mathcal{Q}^\times \) be a primitive homomorphism with character \( \psi: \text{Cl}_P(N_{p}^{a}) \rightarrow \mathcal{Q}^\times \) and \( C(\psi) \) be the conductor of \( \psi \). We write \( \beta \) for the exponent of \( p \) in \( C(\psi) \) and let \( \psi_0 \) denote the primitive character
where

\( \psi_0 : \text{Cl}_F(C(\psi)) \to \overline{\mathbb{Q}}^\times \) associated with \( \psi \). Then we have

\[
\begin{align*}
(12.1a) \quad & |\lambda(T(\mu))|^2 = \mathcal{N}_{F/Q}(\mu)^{[n+2v]+1} \text{ if } 0 < \alpha = \beta, \\
(12.1b) \quad & \lambda(T(\mu))^2 = \mathcal{N}_{F/Q}(\mu)^{[n+2v]} \psi_0(\mu) \quad \text{if } \alpha = 1 \text{ and } \beta = 0, \\
(12.1c) \quad & \lambda(T(\mu)) = 0 \text{ if } \alpha \geq 2 \text{ and } \beta > \alpha,
\end{align*}
\]

where \( n + 2v = [n + 2v]t \) stands for \([n + 2v] \in \mathbb{Z}\).

This fact is well-known; so, we give a sketch of a proof. We shall give a representation theoretic proof. An elementary proof in the case of \( F = \mathbb{Q} \) can be found in [24]. We write \( f_\lambda \) for the unique cusp form in \( S^*_k, w, l(Np^\alpha; M_2(F); C) \) such that \( a(n, f_\lambda) = \lambda(T_0(n)) \) for all \( n \). Let \( \pi(\lambda) \) be the automorphic representation attached to \( f_\lambda \), and let \( \pi_\mu(\lambda) \) be the \( \mu \)-component of \( \pi(\lambda) \). Suppose firstly that \( \pi_\mu(\lambda) \) is a principal series representation \( \pi(\xi, \eta) \) for quasi-characters \( \xi, \eta \) of \( F_\mu^\times \). If \( \xi \) and \( \eta \) are both ramified, then \( \alpha \geq 2 \) and \( \alpha > \beta \) and \( \lambda(T(\mu)) = 0 \). If one of \( \xi \) and \( \eta \), say \( \xi \), is unramified, then \( \xi = \psi_{1/x} \), \( \alpha = \beta \) and \( \lambda(T(\mu)) = \xi(\pi) \) for a prime element \( \pi \) of \( \pi_\mu \). Note that \( \pi_\mu(\lambda) \otimes \omega^a \) for \( a = ([n + 2v] + 1)/2 \) is a unitary representation for \( \omega(x) = [x]_p \). Then, by the classification of unitary representations of \( \text{GL}_2(F_\mu) \), \( \xi \omega^a \) is a unitary character, or otherwise, \( \eta \omega^a \) is also unramified. The latter case cannot happen when \( \alpha > 0 \). Then \( |\lambda(T(\mu))|^2 = |\xi(\pi)|^2 = \mathcal{N}_{F/Q}(\mu)^{[n+2v]+1} \). Secondly, we suppose that \( \pi_\mu(\lambda) \) is a special representation \( \sigma(\xi, \eta) \). If \( \xi \) is ramified, then \( \eta \) is also ramified, \( \alpha \geq 2 \), \( \alpha > \beta \) and \( \lambda(T(\mu)) = 0 \). If \( \xi \) is unramified, then as above, we know that \( \alpha = 1 \), \( \beta = 0 \) and (12.1b) holds. When \( \pi_\mu(\lambda) \) is absolutely cuspidal, then \( \alpha \geq 2 \), \( \alpha > \beta \) and \( \lambda(T(\mu)) = 0 \). This shows the result.

For each ideal \( \mathfrak{m} \) of \( \mathfrak{r} \), we can define a linear map

\[
[\mathfrak{m}] : S^*_k, w, l(N; M_2(F); C) \to S^*_k, w, l(N_{\mathfrak{m}}; M_2(F); C)
\]

by \( a(n, f[\mathfrak{m}]) = a(n_{\mathfrak{m}}^{-1}, f) \) (cf. [23]).

Then we see from (4.2) that \( T_0(\mu) \circ [\mu] \) is the identity map on \( S^*_k, w, l(Np^\alpha; C) \) for \( \alpha > 0 \). For each \( \mathfrak{o} \)-algebra homomorphism \( \lambda : \mathbf{k}_k, w, (Np^\alpha; \mathfrak{o}) \to \overline{\mathbb{Q}}_p \), we put for any field extension \( L/K \)

\[
S^\text{ord}(Np^\alpha, \lambda; L) = \left\{ g \in eS^*_k, w, l(Np^\alpha; M_2(F); L) | g|T_0(\ell) = \lambda(T_0(\ell))g \right\}
\]

except for finitely many prime ideals \( \ell \)

and let \( f_\lambda \) denote the cusp form in \( S^*_k, w, l(Np^\alpha; M_2(F); C) \) satisfying \( a(n, f_\lambda) = \)
\( \lambda(T_0(\mathfrak{n})) \) for all \( \mathfrak{n} \in \Pi(1) \). Then the same argument which proves [11, I, Prop. 4.4 and Lemma 3.3] combined with Lemma 12.2 shows:

**Proposition 12.3.** Suppose that \( N \) is prime to \( p \) and \( k \geq 2t \). Let \( \lambda: A_{k, w}(Np^\alpha; \mathfrak{O}) \to \mathfrak{O} \) be an \( \mathfrak{O} \)-algebra homomorphism of conductor \( C \) and \( \lambda_0: \eta_{k, w}(C; \mathfrak{O}) \to \mathfrak{O} \) be the primitive homomorphism associated with \( \lambda \). Decompose \( C = C_0C_1 \) such that \( C_1 \) divides \( p^\alpha \) and \( C_0 \) divides \( N \). Then \( f = f_{\lambda_0}|_\mathfrak{O} \) is non-zero and \( S^\text{ord}(Np^\alpha, \lambda; K) = \sum_{e|N/C_0} K \cdot (f[[e]]) \). Moreover, let \( V = \bigoplus_{\lambda} S^\text{ord}(Np^\alpha, \lambda; \overline{Q}_p) \), where the sum is taken over all \( \mathfrak{O} \)-algebra homomorphisms \( \lambda: A_{k, w}(Np^\alpha; \mathfrak{O}) \to \overline{Q}_p \) with conductor divisible by \( N \). Then the subalgebra \( \mathcal{H}_V \) of \( \text{End}_\mathbb{Q}_p(V) \) generated over \( K \) by Hecke operators \( T_0(\mathfrak{n}) \) for all \( \mathfrak{n} \) is semi-simple.

We say that \( \lambda: A_{k, w}(Np^\alpha; \mathfrak{O}) \to \overline{Q}_p \) is \( p \)-adically primitive if \( S^\text{ord}(Np^\alpha, \lambda; \overline{Q}_p) \) is of dimension 1 (or equivalently, the conductor of \( \lambda \) is divisible by \( N \)). This notion coincides with the primitivity of complex cusp forms if the conductor of \( \lambda \) is divisible by every prime factor of \( p \). Put \( S_v(N; K/\mathfrak{O}) = S_v^\text{ord}(N; K)/S_v^\text{ord}(N; \mathfrak{O}) \cong S_v^\text{ord}(N; \mathfrak{O}) \otimes_\mathfrak{O} K/\mathfrak{O} \). Then the following fact easily follows from Theorem 5.3:

**Lemma 12.4.** The pairing \( \langle \cdot , \cdot \rangle: h_v^\text{ord}(N; \mathfrak{O}) \times S_v^\text{ord}(N; K/\mathfrak{O}) \to K/\mathfrak{O} \) induced by the pairing of Theorem 5.3 gives isomorphisms:

\[
\begin{align*}
h_v^\text{ord}(N; \mathfrak{O}) &\cong \text{Hom}_c\left(S_v^\text{ord}(N; K/\mathfrak{O}), K/\mathfrak{O}\right), \\
S_v^\text{ord}(N; K/\mathfrak{O}) &\cong \text{Hom}_c\left(h_v^\text{ord}(N; \mathfrak{O}), K/\mathfrak{O}\right).
\end{align*}
\]

**Proof of Theorem 3.6 and Corollary 3.7.** For each divisor \( D \) of \( N \), the map \([D]: S_{k, w, l}(Np^\infty; \mathfrak{O}) \to S_{k, w, l}(Np^\infty; \mathfrak{O})\) preserves the norm (5.3), and since \( D \) is prime to \( p \), \([D]\) commutes with \( T_0(p) \) and hence with \( e \). Therefore, it induces a map \([D]: S_v^\text{ord}(N/D; \mathfrak{O}) \to S_v^\text{ord}(N; \mathfrak{O})\) by continuity and also an injection \([D]: S_v^\text{ord}(N/D; K/\mathfrak{O}) \to S_v^\text{ord}(N; K/\mathfrak{O})\). Then, by Lemma 12.4, it gives a surjective morphism of \( \Lambda \)-modules \([D]^*: h_v^\text{ord}(N; \mathfrak{O}) \to h_v^\text{ord}(N/D; \mathfrak{O})\). The natural inclusion \( I_D: S_v^\text{ord}(N/D; K/\mathfrak{O}) \to S_v^\text{ord}(N; K/\mathfrak{O})\) induces a surjective morphism of \( \Lambda \)-modules \( I_D^*: h_v^\text{ord}(N; \mathfrak{O}) \to h_v^\text{ord}(D; \mathfrak{O})\). Put \( P(N) = P_c(N; \mathfrak{O}) = \cap_{D \supseteq N} (\text{Ker}([D]^*) \cap \text{Ker}(I_D^*))\), and write \( h(N) \) for \( h_v^\text{ord}(N; \mathfrak{O}) \) for simplicity. Put

\[
S(A) = \sum_{[D] \supseteq N} \left(S_v^\text{ord}(N/D; A)[[D]] + S_v^\text{ord}(N/D; A)\right) \quad \text{for} \; A = \mathfrak{O} \text{ and } K/\mathfrak{O}.
\]

Then \( S(A) \) is stable under Hecke operators \( T_0(\mathfrak{n}) \) for all \( \mathfrak{n} \), and therefore, the \( \Lambda \)-module \( P(N) \) is in fact an ideal of \( h(N) \), since \( P(N) \) is the annihilator of \( S(K/\mathfrak{O}) \). Since \( P(N) \) is the Pontryagin dual module of \( \Lambda \)-divisible module
$S_v^{\text{ord}}(N; K/\mathcal{O})/S(K/\mathcal{O})$, it is without $\Lambda$-torsion. Suppose that $h \in \mathcal{P}(N)$ is nilpotent. Then the image $h_\alpha$ in $\mathcal{H}^{\text{ord}}_{k, w}(Np^\alpha; \mathcal{O})$ for every $\alpha > 0$ and every $n - 2v (n \geq 0)$ is also nilpotent. Since $h$ annihilates $S(\mathcal{O})$, $h_\alpha$ annihilates old forms in $S^{\ast}_{k, w, l}(Np^\alpha; M_2(F); K)$. Thus $h_\alpha$ must vanish by Proposition 12.3. This shows that $h = 0$ and thus $\mathcal{P}(N)$ has no nilpotent elements. Now consider $\mathfrak{L} = \mathfrak{L}(N) = h(N) \otimes_\Lambda \mathcal{L}$ for the quotient field $\mathcal{L}$ of $\Lambda$, which is a finite dimensional artinian algebra over $\mathcal{L}$. Since $\mathcal{P}(N) = \mathcal{P}(N) \otimes_\Lambda \mathcal{L}$ has no nilpotent elements, it is actually a semi-simple subalgebra which is a direct factor of $\mathfrak{L}$. Thus we can decompose

\[(12.2a) \quad \mathfrak{L}(N) = \mathcal{P}(N) \oplus \mathcal{B}(N) \quad \text{as an algebra direct sum.}\]

Now we choose an $\mathcal{O}$-valued point $P$ of Spec($\Lambda$) in $\mathfrak{X}_{\text{alg}}(\Lambda)$ with $n(P) \geq 2v$. Considering $P$ as an ideal of $\Lambda$, we write $M_P$ for the localization at $P$ of each $\Lambda$-module $M$ and put $M[P] = \{m \in M | am = 0 \text{ for all } a \in P \}$. Since we know that

\[h(N)_P/PH(N)_P \cong \mathcal{H}^{\text{ord}}_{k, w}(Np^\alpha P, \varepsilon_P; K) \]

\[(k = n(P) - 2v + 2t, w = v + k - t) \quad \text{by Theorem 3.4},\]

the kernel of the natural surjection $\rho: h(N)/PH(N) \to \mathcal{H}^{\text{ord}}_{k, w}(Np^\alpha P, \varepsilon_P; \mathcal{O})$ is annihilated by $\mathcal{O} \in \Lambda - P$. Then $P + \mathcal{O} \Lambda$ contains a power $m^\beta$ of the maximal ideal $m$ of $\Lambda$. Thus $p^\beta$ annihilates Ker($\rho$). By Theorem 5.3, this implies

\[(12.2b) \quad S^{\text{ord}}_v(N; \mathcal{O})[P] = S^{\text{ord}}_{k, w}(Np^\alpha P, \varepsilon_P; \mathcal{O}).\]

Similarly we know

\[(12.2c) \quad S(\mathcal{O})[P] = \sum_{\mathfrak{d} \supset D \supset \mathfrak{N}} \left( S^{\text{ord}}_{k, w}(Np^\alpha P/D, \varepsilon_P; \mathcal{O}) \right) [D] \]

\[+ S^{\text{ord}}_{k, w}(Np^\alpha P/D, \varepsilon_P; \mathcal{O}) \right).\]

Let $V = \bigoplus \lambda S^{\text{ord}}(Np^\alpha P, \lambda; \overline{\mathcal{O}}_p)$, where $\lambda$ runs over all $\mathcal{O}$-algebra homomorphisms: $\mathcal{H}^{\text{ord}}_{k, w}(Np^\alpha P, \varepsilon_P; \mathcal{O}) \to \overline{\mathcal{O}}_p$ with conductor divisible by $N$, and let $\mathcal{H}_V$ be the Hecke algebra for $V$ as in Proposition 12.3. Then (12.2c) implies

\[(12.2d) \quad \mathcal{H}_V \cong \mathcal{P}(N)_P/PP(N)_P, \quad \text{which is an algebra direct factor of}\]

\[\mathcal{H}^{\text{ord}}_{k, w}(Np^\alpha P, \varepsilon_P; K).\]

This implies that the idempotent of $\mathfrak{L}(N)$ is in fact contained in $\mathcal{P}(N)_P$ and thus

\[(12.2e) \quad h(N)_P = \mathcal{P}(N)_P \oplus \mathcal{B}(N)_P \quad \text{for the complementary algebra direct summand} \mathcal{B}(N)_P.\]

We now define two divisors $C, D$ of $N$ with $CD \supset N$, a map

\[i_{C, D}: h(N)_P \longrightarrow P(D)_P \]

by the combination:

\[h(N)_P \xrightarrow{[C]^*} h(N/C)_P \xrightarrow{I_0^*} h(D)_P \xrightarrow{pr} P(D)_P,\]
where \( pr \) denote the projection map of the decomposition (12.2e). We consider the map

\[
i = \bigoplus_{C,D} i_{C,D} : h(N)_p \to \bigoplus_{D[N]} \bigoplus_{C[(N/D)]} P(D)_p,
\]

where \((C, D)\) runs over all pairs of divisors of \( N \) with \( CD \supset N \). By (12.2d), Proposition 12.3 shows that \( i \) induces an isomorphism: \( h(N)_p/PH(N)_p = \bigoplus_{D[N]} \bigoplus_{C[(N/D)]} P(D)_p/PP(D)_p \). Therefore \( i \) induces an isomorphism: \( h(N)_p = \bigoplus_{D} \bigoplus_{C} P(D)_p \). Then the set of \( p \)-adically primitive homomorphisms: \( \mathcal{H}_{k,w}^{\text{ord}}(N^{P}_p, \epsilon_f; \emptyset) \to \overline{Q}_p \) consists of homomorphisms factoring through \( P(N)_p/PP(N)_p \cong \mathcal{H}_Y \). Thus we have a bijection:

\[
\text{Hom}_A(h(N)_p, \overline{\mathcal{P}}) \cong \text{Hom}_K(\mathcal{H}_{k,w}^{\text{ord}}(N^{P}_p, \epsilon_f; K), \overline{Q}_p),
\]

and the set of primitive homomorphisms: \( h^{\text{ord}}_v(N; \emptyset) \to \overline{\mathcal{P}} \) consists of \( \Lambda \)-algebra homomorphisms factoring through \( P(N)_p \) and corresponding to the set of \( p \)-adically primitive homomorphisms of \( \mathcal{H}_{k,w}^{\text{ord}}(N^{P}_p, \epsilon_f; K) \) into \( \overline{Q}_p \). This shows the theorem in view of [23]. Corollary 3.7 also follows from the above proof (especially (12.2e)).

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References


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