Prof. Dr. Gebhard Böckle, Dr. Patrik Hubschmid

Working group seminar WS 2012/13

Transcendence theory in positive characteristic

Wednesdays from 9:15 to 10:45, INF 368, room 248

In this seminar we will study some recent developments in transcendence theory in positive characteristic. The first important goal is to understand Papanikolas' theory which relates the transcendence degree of the period matrix of a rigid analytically trivial dual t-motive to the dimension of its Tannakian Galois group. This important equality can be seen as a function field analogue of Grothendieck's period conjecture for abelian varieties. Unlike in the function field case, Grothendieck's conjecture is widely open in the number field case. For an elliptic curve E over $\overline{\mathbb{Q}}$ with periods ω_1, ω_2 and corresponding quasi-periods η_1, η_2, it states that

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}(\omega_1, \omega_2, \eta_1, \eta_2) = \begin{cases} 4 & \text{if } E \text{ has no complex multiplication,} \\ 2 & \text{if } E \text{ has complex multiplication.} \end{cases}$$

This is known in the case of complex multiplication by work of Chudnovsky, but open in the non-CM case where one only knows that the periods and quasi-periods are linearly independent over $\overline{\mathbb{Q}}$ by work of Masser. The analogous result for rank 2 Drinfeld modules will be proved in talk 13 and is due to Chang-Papanikolas.

Before studying Papanikolas' theory we will cover in detail the formalism of neutral Tannakian categories. The rest of the seminar is devoted to the following applications of Papanikolas' theory:

- Algebraic independence of Carlitz logarithms of algebraic numbers
- Algebraic relations among Carlitz zeta values
- Algebraic independence of periods, quasi-periods and logarithms of Drinfeld modules

1 Introduction to t-modules and t-motives, part I

The aim of this lecture is to introduce t-modules and t-motives and give some examples of t-modules following the introductory article [BP11] by Brownawell and Papanikolas. The talk should contain the following parts:

- Definition of t-modules, morphisms between t-modules and sub-t-modules
- Exponential function of t-modules over $\mathbb{C}_\infty$, sketch of the proof of the existence of the exponential function following [And86, §2].
- Uniformizable t-modules
- Examples of t-modules: Carlitz module, Drinfeld modules, tensor powers of the Carlitz module, t-modules arising from quasi-periodic functions

- Definition of t-motives, abelian t-motives, t-motives associated to t-modules, equivalence of the categories of abelian t-modules and abelian t-motives (without proof)

Literature: [BP11, p. 8–15], [And86, §1, 2.1, 2.2]
Level of difficulty: easy-moderate
Speaker: Carolin Peternell 17.10.

2 Introduction to t-modules and t-motives, part II

This lecture should be the continuation of the introduction to t-modules and t-motives starting after theorem 4.1.1 in [BP11]. The talk should contain the following parts:

- Definition of *pure* t-motives, the weight of pure t-motives and the tensor product of pure t-motives following sections 1.9-1.11 in [And86]
- Definition of *rigid analytically trivial* t-motives, mention without proof theorem 4.1.2
- t-motives associated to Drinfeld modules of rank r: Explain that they are pure of rank r and weight $1/r$.
- Example: t-motive associated to tensor powers of the Carlitz module
- Definition of (abelian) dual t-motives, dual t-motives associated to t-modules, equivalence of the categories of abelian t-modules and abelian dual t-motives (without proof), rigid analytic triviality of dual t-motives
- Example: Rigid analytic trivialization of the dual t-motive associated to the Carlitz module.
- Give an explicit proof of the rigid analytic triviality of the dual t-motive associated to a Drinfeld module of rank r using Anderson generating functions following [CP12, §3.4] or [Pel08, §4]. Discuss in particular in the rank 2 case the relation of the rigid analytic trivialization to the periods and quasi-periods of the original Drinfeld module. See [CP12, §3.1] for an explanation of periods, quasi-periods and period matrices of Drinfeld modules.

Literature: [BP11, p. 15-21], [And86, §1.9-1.11], [CP12, §3.1, 3.4], [Pel08, §4]
Level of difficulty: easy-moderate
Speaker: Yujia Qiu 24.10.

3 ABP-criterion

The aim of this talk is to explain the proof of the linear independence criterion introduced by Anderson, Brownawell and Papanikolas which is the basis for all algebraic independence statements in this seminar following section 3 of [ABP04]. First formulate the criterion and then explain how it can be used to prove that the fundamental period of the Carlitz module is transcendental. Before going through the proof in [ABP04, §3.3, 3.4], explain shortly the proof in the one-dimensional case. Mention that lemma 3.3.5 is an analogue of the classical Thue-Siegel lemma which was also proved using the pigeonhole principle. Proposition 3.1.3 in [ABP04] can be skipped because it will be discussed in talk 9.
Literature: [ABP04, §3], [BP08, §4.4]
Level of difficulty: easy-moderate
Speaker: 31.10.

4 Tensor categories

In this talk, abstract tensor categories should be discussed following chapter 1 of [DM12]. The talk should at least contain the following parts:

- axioms defining a tensor category
- tensor product of finitely many objects (state proposition 1.5 without proof)
- internal Hom and duals
- rigid tensor categories
- tensor functors
- Some examples of tensor categories

Level of difficulty: easy-moderate
Literature: [DM12, chapter 1]
Speaker: Yamidt Bermudez Tobón 7.11.

5 Neutral Tannakian categories

The aim of this talk is to formulate and prove the central result that each neutral Tannakian category over a field k is equivalent to the category of finite dimensional representations of some affine group scheme over k. Follow the exposition in chapter 2 of [DM12]. First recall the basic theory of affine group schemes and their representations (p. 17-20) and then formulate and prove the main theorem 2.11. Discuss at least the examples 2.30 and 2.31. Example 2.30 will be later used to compute the Galois group of the Carlitz motive. Finally explain propositions 2.20 (b) and 2.21 which will be used in talk 8.
Level of difficulty: moderate
Background: algebraic groups, algebraic geometry
Literature: [DM12, chapter 2]
Speaker: 14.11.

6 Tannakian categories of t-motives

This talk should introduce a neutral Tannakian category $\mathcal{T}$ over $\mathbb{F}_q(t)$ which contains the category of rigid analytically trivial dual t-motives up to isogeny as a full subcategory.
Start with the definition of the category of pre-t-motives and show that it is a rigid abelian $\mathbb{F}_q(t)$-linear tensor category following the exposition in subsection 3.2 in [Pap08]. Then define the subcategory $\mathcal{R}$ of rigid analytically trivial pre-t-motives and show that it is a neutral Tannakian category. Finally explain that there is a fully faithful functor from the category of rigid analytically trivial dual t-motives up to isogeny to $\mathcal{R}$ and define the Tannakian subcategory $\mathcal{T}$ of $\mathcal{R}$ generated by the image of this functor. Proposition 3.3.9(c) can be skipped because we will cover it in talk 9 and we will not use it before.
Level of difficulty: easy-moderate
Background: Tannakian formalism (talks 4 and 5)
Literature: [Pap08, §3.1-3.4]
Speaker: 21.11.

7 Galois groups of σ-semilinear equations

Consider fields $F \subset K \subset L$ together with an automorphism $\sigma : L \to L$ which restricts to an automorphism of F and K such that $L^\sigma = F$ and L is a separable (not necessarily algebraic) extension of K. The goal of this talk is to assign a *difference Galois group* to a fundamental system of solutions $\psi \in \mathrm{Mat}_{r\times 1}(L)$ of σ-semilinear equations

$$\sigma(\psi) = \Phi\psi$$

associated to a fixed matrix $\Phi \in \mathrm{GL}_r(K)$. The difference Galois group can be realized as a closed subgroup scheme of $\mathrm{GL}_{r/F}$.

Explain the construction of the difference Galois group following the exposition in [Pap08, §4.1-4.2]. Finally explain the criterion for smoothness of the Galois group in [Pap08, §4.3].

Level of difficulty: moderate

Background: algebraic geometry

Literature: [Pap08, §4.1-4.3]

Speaker: 28.11.

8 Galois groups of t-motives

Every object M of the category $\mathcal{T}$ defined in talk 6 generates a Tannakian subcategory of $\mathcal{T}$. Hence, by Tannakian duality, one can associate a Tannakian Galois group to M which is a linear algebraic group over $\mathbb{F}_q(t)$. The aim of this talk is to show that this Galois group is isomorphic to the Galois group associated to a rigid analytic trivializiation of M in the previous talk.

Start with the definition of the Tannakian Galois group associated to an object of $\mathcal{T}$ and the example of the Carlitz motive in [Pap08, §3.5]. Continue with a sketch of the proof of [Pap08, Theorem 4.4.6] and then follow the exposition in [Pap08, §4.5].

Level of difficulty: moderate-difficult

Background: algebraic geometry, Tannakian formalism (talks 4 and 5), Tannakian categories of t-motives (talk 6)

Literature: [Pap08, §3.5, 4.4-4.5]

Speaker: 5.12.

9 Transcendence degrees of period matrices

The goal of this talk is to prove the main theorem of [Pap08] which relates the transcendence degree of the period matrix of an object in $\mathcal{T}$ to the dimension of its Tannakian Galois group.

Start with proposition 3.3.9.(c) in [Pap08] which shows the existence of a rigid analytic trivialization with matrix entries in the ring of restricted power series (converging on the closed unit disk). Continue with a sketch of the proof of Proposition 3.1.3 of [ABP04], which shows that these matrix entries even lie in the ring of entire power series. Then proceed with the proof of the main theorem of [Pap08] using the ABP-criterion from talk 3.

Level of difficulty: moderate-difficult

Background: algebraic geometry, previous talks on Papanikolas' theory

Literature: [Pap08, §5.1-5.2, proposition 3.3.9(c)], [ABP04, proposition 3.1.3]

Speaker: 12.12.

10 Algebraic independence of Carlitz logarithms

The aim of this talk is Papanikolas' transcendence result stating that Carlitz logarithms of elements in $\overline{\mathbb{F}_q(\theta)}$ are algebraically independent over $\overline{\mathbb{F}_q(\theta)}$ provided that they are linearly independent over $\mathbb{F}_q(\theta)$.

Follow the exposition in section 6 in [Pap08]. Start with the construction of an object X in $\mathcal{T}$ whose periods (matrix entries of a rigid analytic trivialization evaluated at $t = \theta$) are related to the considered Carlitz logarithms (section 6.1). Then explain how explicit equations for the Galois group of X can be found (section 6.2) and relate the dimension of the Galois group with the dimension of the $\mathbb{F}_q(\theta)$-linear span of the Carlitz period and the considered Carlitz logarithms (section 6.3). Proceed with the proof of the actual result (section 6.4).

Level of difficulty: moderate

Background: Papanikolas' theory (in particular talks 6 and 8)

Literature: [Pap08, §6]

Speaker: 19.12.

11 Transcendence of special Carlitz zeta values

In this talk, we apply Papanikolas' theory to prove the transcendence of special Carlitz zeta values $\zeta_C(n)$ for positive integers n following the article [CY07] by Chang and Yu. This transcendence result was first established by Yu in [Yu91].

Follow section 3 and the beginning of section 4 in [CY07]. After introducing Carlitz polylogarithms, explain the construction of a rigid analytically trivial dual t-motive with given Carlitz polylogarithms as periods in section 3.1. Then state Theorem 3.1 concerning the dimension of the Galois group of this t-motive. Its proof can be left out because it follows closely Papanikolas' ideas presented in talk 10. Continue with the two corollaries in section 3.2 and then conclude the transcendence of $\zeta_C(n)$ using a beautiful formula expressing $\zeta_C(n)$ as a linear combination of n-th Carlitz polylogarithms with rational coefficients due to Anderson-Thakur [AT90, proof of theorem 3.8.3(I)]. Give a sketch of the proof of the latter formula.

Level of difficulty: moderate

Background: Papanikolas' theory (in particular talks 6 and 8)

Literature: [CY07, p. 329-334], [AT90, §3]

Speaker: 9.1.

12 Algebraic relations among special Carlitz zeta values

The aim of this talk is to prove the result of Chang-Yu [CY07] about algebraic relations among the special Carlitz zeta values $\zeta_C(n)$ and the Carlitz period $\tilde{\pi}$. The result states that the Euler-Carlitz relations expressing $\zeta_C(n)$ as a $\mathbb{F}_q(\theta)$-scalar multiple of $\tilde{\pi}^n$ for n divisible by $q-1$ and the obvious p-th power relations $\zeta_C(p^m n) = \zeta_C(n)^{p^m}$ are the only algebraic relations. Start with a sketch of the proof of the Euler-Carlitz relations after [Car35, theorem 9.3]. Then follow the article of Chang and Yu [CY07] starting after corollary 4.3.

Level of difficulty: moderate

Background: Papanikolas' theory (in particular talks 6 and 8)

Literature: [CY07, p. 334-339], [Car35]

Speaker: 16.1.

13 Algebraic independence of periods and quasi-periods of Drinfeld modules

In this talk, we apply Papanikolas' theory to compute the transcendence degree of period matrices of Drinfeld modules over $\overline{\mathbb{F}_q(\theta)}$ and give an explicit description of the Galois group of the dual t-motive associated to a Drinfeld module following section 3 in [CP12].

Start with a reminder about periods and quasi-periods of Drinfeld modules and the explicit description of the rigid analytic trivialization of the dual t-motive associated to a Drinfeld module using Anderson generating functions ([CP12, §3.1, 3.3, 3.4], talk 2). Continue with a description of the Galois representation on the t-adic Tate module of a Drinfeld module in terms of Anderson generating functions ([CP12, §3.2]) and then present the proof of the main result [CP12, theorem 3.5.4] citing a result of Pink [Pin97] on the openness of the image of the Galois representation on the t-adic Tate module of a Drinfeld module without proof.

Level of difficulty: moderate

Background: Papanikolas' theory (in particular talks 6 and 8)

Literature: [CP12, §3]

Speaker: 23.1.

14 Algebraic independence of Drinfeld logarithms

The goal of this talk is to go through the proof of the algebraic independence of Drinfeld quasi-logarithms of algebraic numbers due to Chang-Papanikolas [CP12].

Explain first how one can associate to a Drinfeld logarithm a rigid analytically dual t-motive which is an extension of the considered Drinfeld module by the identity object of the category of pre-t-motives (section 4 of [CP12]). Then go in detail through section 5.1 of [CP12] to prove algebraic independence of Drinfeld quasi-logarithms in the case when A is a polynomial ring. If you are short of time, skip the proof of lemma 5.1.3 completely, otherwise just explain how it boils down to a question on algebraic groups.

Level of difficulty: moderate-difficult

Background: Algebraic geometry, Papanikolas' theory (in particular talks 6 and 8)

Literature: [CP12, §4, 5.1]

Speaker: 30.1.

Back-up date: 6.2.

References

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[Yu91] Jing Yu. Transcendence and special zeta values in characteristic p. *Ann. of Math. (2)*, 134(1):1–23, 1991.