Modularity lifting theorems

Wednesday, IWR 368, Room 248, 9 c.t.

Prof. Dr. Gebhard Böckle
Dr. Juan Marcos Cerviño
Konrad Fischer

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The main **aim** of the seminar is the following modularity lifting theorem:

Theorem 1. *Let $F/\mathbb{Q}$ be a totally real number field and let $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation of its Galois group, where $p > 5$. Assume that ρ satisfies the following assumptions:*

a) ρ is unramified outside a finite set of places.
b) ρ is odd, i.e. $\det(\rho(c)) = -1$ for any complex conjugation c.
c) ρ is potentially crystalline and ordinary at all places v dividing p.
d) $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.
e) There is a parallel weight 2 Hilbert modular form f such that the associated ρ_f is potentially crystalline and ordinary above p and $\overline{\rho} = \overline{\rho}_f$.

Then there exists a Hilbert modular form g such that $\rho_g \cong \rho$.

The strategy to show such a theorem is strongly influenced by the famous work of Wiles and Taylor on Fermat's last theorem. Namely, they proved the

Modularity theorem ([Wil95], [TW95]): *Every semi-stable elliptic curve over $\mathbb{Q}$ is modular.*

The link between these seemingly distinct statements is the use of Galois representations and in particular their deformation theory, since there are several equivalent formulations of what it means for an elliptic curve $E/\mathbb{Q}$ to be modular: Let $f \in S_k(\Gamma_1(N), \chi)$ be a Hecke cuspform of weight ≥ 1. Let K_f be the field generated over $\mathbb{Q}$ by the Fourier coefficients $\alpha_\ell(f)$ for primes $\ell \nmid N$. Then $K_f \supseteq \mathbb{Q}(\chi)$ is a number field. Let $\lambda \mid p$ be a place of K_f. To this setting by the work of Eichler, Shimura, Deligne and Serre there is attached an abelian variety A_f over K_f and a unique continous representation $\rho_{E,\lambda}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\lambda})$, unramified at $\ell \nmid Np$ and such that $\mathrm{tr}(\rho_{E,\lambda}(\mathrm{Frob}_\ell)) = \alpha_\ell(f)$, $\det(\rho_f) = \chi\epsilon_p^{k-1}$.

Then the following statements are equivalent and we call $E/\mathbb{Q}$ modular if they hold.

(1) E is over $\mathbb{Q}$ isogenous to A_f for some f of some level.
(2) E is over $\mathbb{Q}$ isogenous to A_f for some f of level equal to the conductor of E.
(3) For some prime ℓ the Galois representation $\rho_{E,\ell}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_l)$ on the ℓ-adic Tate module of E is equivalent to ρ_f for some f.
(4) (3) holds for all primes ℓ.

The link to the above modularity lifting theorem is then given by (3) and one is left with verifying the assumptions of Theorem 1 for the class of semi-stable elliptic curves. Especially the residual modularity condition *e*) is intricate and linked to a special case of Serre's conjecture. Fortunately, this special case was known at the time of Wiles by work of Langlands-Tunnel and Ribet.

As mentioned above, the way to prove Theorem 1 is by introducing a deformation ring R and a certain Hecke algebra $\mathbb{T}$. R is constructed such that it 'knows' the representations that reduce to $\overline{\rho}$: They correspond exactly to ring homomorphisms with domain R. $\mathbb{T}$ on the other hand parametrizes the modular Galois representations reducing to $\overline{\rho}$. By assumption e) we get a ring map $R \twoheadrightarrow \mathbb{T}$. If we can show that this is an isomorphism (fittingly called an '$R = \mathbb{T}$ theorem') we are done.

The **outline** of the seminar will roughly be as follows: Recalling classical modular forms and elliptic curves we realize that p-adic Galois representations come up in both settings: Hecke eigenvalues occur as traces of Galois representations and the level of a modular form is linked to ramification phenomena. This makes up the talks **1** and **2**.

We then leave the world of modular forms and first study how to parametrize p-adic representations of given ramification type with fixed reduction mod p. This is a mix of Galois theory and representations. Talk **3** introduces Mazur's deformation theory of Galois representations. The cohomology of the tangent space of the parametrizing 'deformation ring' gives us crucial information on the dimension of this ring. Talks **4** and **5** explain this. Technically the most difficult part is when we try to fix the ramification type. As is to be expected, the places of residue characteristic p are the most difficult ones and we only deal with one special case[1]. This is done in the talks **6-7**.

In talks **8-9** we come back to modular forms in more generality. The language of automorphic forms and representations is explained in the GL_2 and quaternion algebra setting. The Jacquet-Langlands correspondence relates these two. The quaternion forms were not present in Wiles' original work. They are convenient in that they allow to substitute étaleness properties of modular curves by simple group theoretic arguments for functions on finite sets. A further simplification is achieved by making a solvable base change. Talk **11** will state the necessary results.

The last 3 talks will tie together the deformation and Hecke rings by a commutative algebra reasoning, known as the patching argument. Using earlier dimension results, talk **12** constructs a family of auxilliary places that were originally unramified. Allowing ramification at these places corresponds to level raising on the Hecke side. The conditions we impose on these places will ensure that there still is a strong relation to the original level. For deformation rings this is almost immediate, for the modular side this is the aim of talk **13**. Finally, talk **14** proves an $R = \mathbb{T}$ theorem on the original level by a kind of 'horizontal Iwasawa theory' over the family of auxilliary levels.

Talk 1. 16.10. **Algebraic properties of classical Hecke algebras - Speaker: Juan Marcos Cerviño**

First give a short overview of the definitions in classical modular form theory (taking for example §1.2-1.6 of [DDT97] as a guide):

- Modular forms and cusp forms, Modular curves
- Mention, that cusp forms of weight 2 correspond to differentials on the modular curve
- Hecke and diamond operators for forms of level $\Gamma_1(N)$, newforms
- "Motivate" the fact, that the full Hecke algebra $\mathbb{T}_{\mathbb{Z}}$ is module finite over $\mathbb{Z}$ (cf. theorem 1.6 from lec. 8 of [C$^+$10]) by identifying $S_k(\Gamma, \mathbb{Z})$ with group cohomology.

Now let $\mathcal{O}$ be the ring of integers of a p-adic local field K. The main part of this talk are results on algebraic properties of $\mathbb{T}_{\mathcal{O}}$ and its localizations:

- Explain and prove the diagram (4.1.1) of [DDT97]: Galois orbits of normalized eigenforms in $S_k(\Gamma, \overline{K})$ correspond to irreducible components of $\operatorname{Spec} \mathbb{T}_{\mathcal{O}}$ and on each of the latter lies a unique closed point.
- State that to such a closed point there is associated a residual Galois representation.
- Proof theorem 3.1 from lec. 8 of [C$^+$10].

References: lec. 8 in [C$^+$10] *and references therein: For the first part:* [DS05] *or* [Shi71]*; for the second part:* [Mat80]*,* [DDT97] *§4.1 - Prerequisites: Classical modular forms*

Talk 2. 23.10. **Elliptic curves and FLT - Speaker: Konrad Fischer**

This talk should first give the basic definitions considering elliptic curves and their Galois representations. It should then motivate modularity lifting results by outlining the Frey-Serre-Ribet-Wiles-Taylor line of thought.
Elliptic curves: Weierstrass Equation, E.C. over $\mathbb{C}, \mathbb{Q}, \mathbb{Q}_p$, conductor, reduction at p, Tate-Modules, Global and local Galois groups, Galois representations
Overview of FLT: Fermat's Last Theorem, Frey's Curve, Serre's Conjecture (cf. Ribet-Stein in Park City Series, Vol 9), Ribet's level lowering, consequences of $R = T$ theorems, Langlands-Tunnel theorem. The 3-5-trick and why we would like a modularity lifting theorem for $p \geq 3$.
References: §1.1 and the introduction of [DDT97] *or a similar overview,* [Sil92] *for elliptic curves.*
Prerequisites: Elliptic curves

Talk 3. 30.10. **Universal deformation rings - Speaker: Yujia Qiu**

We start with a given 'residual' representation $\overline{\rho}\colon \Gamma \to \mathrm{GL}_n(k)$, where k is a finite field of characteristic p. Let Λ be a noetherian complete dvr with residue field k, e.g. $W(k)$.

- Introduce the categories $\mathcal{C}_\Lambda, \widehat{\mathcal{C}}_\Lambda$ and the deformation functors $D_{\overline{\rho}}, D^{\square}_{\overline{\rho}}$ of $\overline{\rho}$
- Give equivalent statements for the finiteness condition Φ_p and proof that Φ_p holds for (restricted global and local) Galois groups via CFT.

[1]To deduce FLT in a similar manner we would need to understand flat non-ordinary deformation rings. This involves finite flat group schemes and Fontaine-Lafaille theory. We will content ourselves with the ordinary-crystalline case.

- Proof the representability of $D^{\square}_{\overline{\rho}}$
- Mention Schlessingers Criterion: If $\overline{\rho}$ is 'Schur' and Φ_p is satisfied then $D_{\overline{\rho}}$ is representable.
- Later we need Prop. 1.7 in lec. 5 [C+10]: if $\overline{\rho}$ is 'successively nonsplit extension' then $\overline{\rho}$ is Schur[2]
- Introduce $\operatorname{ad}\overline{\rho}$ and the tangent space of a representable deformation functor.
- Mention how the tangent space of a deformation functor is naturally a vector space over k.
- Proof as much as possible of the k-isomorphisms: $\mathrm{H}^1(\Gamma, \operatorname{ad}\overline{\rho}) \cong \mathrm{Ext}^1_{k[\Gamma]}(\overline{\rho}, \overline{\rho}) \cong D_{\overline{\rho}}(k[\varepsilon])$.

References: A possible guideline: Lec. 3 in [C+10]*, references with many details:* [Gou01] *and* [Maz97]*, shorter and modern:* [Böc13]*, §1. - Prerequisites: a bit of group cohomology, the results of class field theory*

Talk 4. 6.11. **Galois cohomology: The Poitou-Tate sequence and dual Selmer groups - Speaker: Katharina Hübner/Johannes Anschütz**
The reason we're interested in Galois cohomology is that it gives certain deformation conditions via the isomorphism from last talk: $\mathrm{H}^1(G_{K,S}, \operatorname{ad}\overline{\rho}) \cong D_{\overline{\rho}}(k[\varepsilon])$.

- Recall continous group cohomology and some functorial properties: Res-Inf-sequence, Long-exact sequence.
- Specify to Galois groups and define local conditions: unramified deformations, Selmer groups, Poitou-Tate-Duality and dual Selmer groups.
- Give the local and global Euler-Poincare-characteristic.
- Proof Wiles' product formula.
- Calculate an example, if time permits.

References: Lec. 7 in [C+10]*,* [Was97],[NSW08] - *Prerequisites: Galois cohomology*

Talk 5. 13.11. **Deformation conditions and Global Deformation rings - Speaker: Andreas Maurischat**
In talk 3 we learned about universal deformation rings. Now we get more number theoretic: If we are deforming a (global) Galois representation and we fix a lift of the residual determinant, can we restrict the determinant of deformations to be this lift? This is an example of a 'global' condition. It turns out such conditions are often representable:

- Different perspectives on deformation conditions: subfunctors, closed subspaces of $\operatorname{Spec} R_{\overline{\rho}}$, relative representability considered by Mazur and Kisin.
- Proof that the following are deformation conditions: fixing the determinant and how this is related to ad^0, being unramified at a given prime, being ordinary at a given prime.

The last two conditions are of a local nature: Is there a ring representing the deformations with a specified ramification type at/outside of a specified set of primes? To impose such conditions at several primes at once we need a tensor product in our category $\widehat{\mathcal{C}}_\Lambda$:

- Define completed tensor products: their universal property, some examples: Noetherianness can fail, if not completed.
- Show that the global deformation ring naturally becomes an algebra over the local ones.
- If time permits: some easy relative dimension bounds.

References: For the first part: Lec. 5 in [C+10], [Gou01], [Böc13]*; for the second part: Lec. 6 of* [C+10]
Prerequisites: Talk 3

Talk 6. 20.11. **Local deformation rings I: away from p - Speaker: Gebhard Böckle**
To show that certain functors are smooth and connected we do many tangent space(=Galois cohomology) calculations. Knowledge about formal schemes and rigid spaces is also helpful.

- Categories fibered in groupoids ([Böc13], section 1.6)
- Facts without proofs about generic fibers of deformation rings (see also lec. 3 in [C+10]): Lem. 3.1.1 and Theorem 3.1.2 in [Böc13]
- State Grothendieck's monodromy theorem and define *inertial WD-types.*

[2]there should be an easier proof of this

- Proof as much as possible from theorem 3.3.1 in [Böc13]; at least, that $\operatorname{Spf} R^{\psi,\square}_{\overline{\rho}}[\frac{1}{p}]$ is 3-dimensional with smooth components if we assume representability and smoothness of the Steinberg components.
- Proof smoothness of the unramified component, assuming existence (§4 in lec. 20 of [C+10]).
- Kisin's proof of representability involves a geometric argument: One includes the datum of a line to the deformations and hence a $\mathbb{P}^1$ appears. Explain some of this following 3.5 in [Böc13].

References: lec. 20 in [C+10]*, lecture 3 in* [Böc13]*; the original source is Kisin* [Kis09b]; maybe easier is Gee [Gee11]
Prerequisites: Weil-Deligne-Representations; Algebraic Geometry: formal schemes, projective morphisms

Talk 7. 27.11. **Local deformation rings II: above p - Speaker: Ann-Kristin Juschka**
The easiest condition to impose at places above p (and very similar in technique to the Steinberg case in the previous talk) is *ordinarity* for weight 2. This talk deals with the ordinary crystalline deformation rings at p. Fortunately p-adic Hodge theory does not play a role here. The main existence result is in §3. Later talks will refer in particular to 4.3, 4.5 and 4.6.

- §1, Motivation: Introduce the generic fiber R_E, MaxSpec(R_E). Mention: R_E is Jacobson (cf. talk 3). Points of the latter correspond to E-algebra representations ρ_x. We want to understand the **P**-loci[3].
- §2, AlgGeo: The two reductions $X_0, \overline{X}$ of proper $X \to \operatorname{Spec} R$ coming from $\mathfrak{m}_R$ and π. Introduce $(*)$: all completed local rings of X at closed points are smooth. Mention Prop. 2.1 and deduce Lemma 2.2 from it: the components of X_E and X_0 correspond. There is a functorial criterion for $(*)$: Prop. 2.3. (formally smoothness for artinian points). State the criterion for $X_E \to R_E$ to be a closed immersion.
- §3, Ord-Cryst defo's: Explain how ordinary E'-valued ρ correspond to $\mathrm{H}^1(I_K, \mathbb{Z}_p(1)) \otimes \Lambda'$. ρ is crystalline if this class lies in a certain hyperplane. In general we need a topology-free definition: Def. 3.5., beware that here G_K-lines may be not unique! H^1_{crys} is compatible with direct limits and compares with the above definition. Mention lemma 3.10 (there are two proofs but explain none). Explain as much as possible from the proof of Theorem 3.11: There is a closed X in $\mathbb{P}^1_R$ whose infinitesimal fibers classify ord-cryst structures.
- §4, Properties of X: Show: X is regular and flat using §2 and Lemma 3.10. State Prop. 4.2 without proof. State 4.3-4.6, proving as much as time permits: 4.6 has an easy comm. alg. proof, while the proof of 4.5 is longer but uses many different ideas.

References: Part 1: lec. 20 in [C+10] *and §3.7 in* [Böc13] *(also with some background on formal schemes) following Kisin* [Kis09a]
Prerequisites: Algebraic Geometry (e.g. properness, smoothness, line bundles), Galois representations and cohomology

Talk 8. 4.12. **Automorphic representations - Speaker: Kathrin Maurischat/Mirko Rösner**
Since one could easily hold several seminars about this topic alone, the emphasis in this talk should lie on clear presentation of a few basic properties and examples, rather then sketchy proofs and definitions in utmost generality.
Part 1 - Local theory:

- Recall admissible, smooth etc. complex representations of p-adic groups.
- Recall induction from subgroups, example: classification of $GL_2(\mathbb{Q}_p)$-representations.
- The Hecke algebra $H(G,K) := \mathcal{C}^\infty(K\backslash G/K)$, K-spherical representations, examples: $K = \mathrm{GL}_2(\mathcal{O})$, $K =$ Iwahori.

Part 2 - Global theory:

- To have a class of important examples for the global theory, first introduce quaternion algebras (QA): Local/global QA, ramification set, classification of QA by even ramification sets, orders in QA.
- Then explain automorphic forms with this examples in mind: Automorphic forms and representations for GL_2 and its inner forms over totally real number fields (being very brief at the archimedean places[4]) The next talk will expand on this example.
- Mention the tensor product theorem.

[3] **P** being here just 'ord-cryst'
[4] one could for example 'only consider groups G, such that $G_{v|\infty} \cong \mathrm{GL}_2(\mathbb{R})$ or $\mathbb{H}^\times$' instead of defining $(\mathfrak{g}, K)$-modules etc.

References: Bushnell-Henniart [BH] *and Bump* [Bum97] *for the first part, Borel-Jacquet's article from Corvallis* [BJ79] *for the automorphic forms/reps part, Vigneras in french or Janntzen-Schwermer for the quaternion part; also lecs. 9+10 in* [C$^+$10] *for inspiration on the presentation*
Prerequisites: Some familiarity with the representation theory of GL_2

Talk 9. 11.12. **Hilbert and quaternionic modular forms: Speaker: Patrik Hubschmid**
The emphasis of the talk should be on the last part (possibly leaving out the middle on $\mathrm{GL}_{2/F}$), explaining levels and Hecke operators for quaternionic modular forms and making clear the role of strong approximation.

- How is a classical modular form adelized? A short sketch for $\mathrm{GL}_2(\mathbb{Q})$ is in [Gee13], ex. 4.9. Longer in [Bum97], §3.6.
- In the adelic language the generalization to Hilbert modular forms for GL_2 comes natural, e.g. [Dim13], §1.
- Finally, we will need automorphic forms for quaternion algebras over totally real fields: As for GL_2 there is the classical point of view (cf. [DV13], §3) and the adelic one (cf. [DV13], §7). Define their levels (orders) and Hecke operators.
- Example: What happens for totally definite quaternion algebras D? By strong approximation we get functions on a finite set! - [Tay06], §1.
- If time permits: What if a QA is split at exactly one infinite place? Shimura curves!

References: Articles from [Dar13] - *Prerequisites: Adeles, quaternion algebras, modular forms*

Talk 10. 18.12. **Reserve date** and **Overview of the things to come**
At this point we will probably be behind schedule. If there is time, perhaps someone could cover some of the following:

- General fibers of deformation rings and field valued p-adic representations.
- Why do we only need an $R[\frac{1}{p}] = T[\frac{1}{p}]$-theorem?
- Structure of the patching argument.

References: Lectures 16 and 18 from [C$^+$10]

Talk 11. 8.1. **Langlands base change and the Jacquet-Langlands correspondence - Speaker: Tommaso Centeleghe**
We will use solvable base change (proved by trace formula techniques) to simplify our setting for modularity lifting. This talk will likely contain no proofs, but many beautiful theorems and hopefully some examples.

- Meta-Plan: Explain the results in the title and link them to potential automorphy. For example following 4.17-4.26 of [Gee13].
- An exact statement of the Jacquet-Langlands correspondance is for example in Taylors thesis [Tay89], p.270-271. We need a result as in §1.5 of lec. 16 in [C$^+$10].
- For base change see lec. 15 in [C$^+$10] and [GL79].

References: Lecs. 15+16 of [C$^+$10], *Gee* [Gee13], *Taylor* [Tay89], *the overview in Corvallis* [GL79],
Prerequisites: Some familiarity with the Langlands program

Talk 12. 15.1. **Existence of Taylor-Wiles systems - Speaker: David Guiraud**
By solvable base change we can assume that $\overline{\rho} = \overline{\rho}_f$ with modular ρ_f which is everywhere Steinberg where it is ramified outside $p\infty$. This talk constructs by Galois cohomology methods a family of sets of places for the patching argument.

- Define Taylor-Wiles places of level $n \geq 1$.
- Show that the action of the ramification group at a T-W-place is via the sum of two tame characters - Lemma 2 in [dS97] or lec. 24 in [C$^+$10]. This gives $R_Q^\square$ the structure of an $\mathcal{O}[\Delta_Q]$-algebra. Introduce the augmentation ideal $\mathfrak{a}_Q$ and (if time permits) show $R_Q^\square/\mathfrak{a}_Q \cong R_\emptyset^\square$.
- Mention that by easy group theory the so-called Taylor-Wiles condition d) form Theorem 1 above implies that $\overline{\rho}|_{G_{F(\zeta_{p^n})}}$ is absolutely irreducible for $n \geq 1$.

- Proof the first main theorem: Existence of T-W sets of arbitrary level. This involves some case by case studies of subgroups of PGL_2: don't do all of them!
- Proof the second theorem: The number of local generators for the global deformation ring is constant over the various T-W-levels.

References: lec. 24 in [C$^+$10]*, de Shalit* [dS97] *for some details - Prerequisites: talk 4 and 5*

Talk 13. 22.1. **Construction of Hecke modules for patching - Speaker: Sundeep Balaji**
Here the modules of quaternionic automorphic forms $M_{Q_n}, n \geq 0$ are introduced and several key properties established: they are free of rank 2 over a suitable Hecke algebra and related to the $n = 0$ level by the augmentation ideal $\mathfrak{a}_n$ from last talk. The outline for this talk is lec. 25 from [C$^+$10].

- Proof the canonical isomorphism $R_Q^\square/\mathfrak{a}_Q \overset{(*)}{\cong} R_\emptyset^\square$, cf. Lemma 2.0.2 in lec. 25 of [C$^+$10], if this was not done in the last talk.

We want to construct $\mathcal{O}[\Delta_Q]$-modules M_Q for every T-W set Q, such that they satisfy the analogue of $(*)$:

- Recall the space of quaternionic automorphic forms, $S(U)$ of level $U \subseteq (D \otimes \mathbb{A}_F^f)^\times$. We have already seen them in talk 9. Define the 'good' (=prime-to-the-level) Hecke algebra $\mathbb{T}(U)$.
- Define levels U_Q, V_Q, such that $V_Q/U_Q \cong \Delta_Q$ and their 'Q-enriched' Hecke algebras $\mathbb{T}^+$, following lec. 25 from [C$^+$10]
- Explain the 'smallness condition' and why we can choose an auxilliary place v_{aux}.
- Apply elementary group theoretic lemmata to our setting: $S[U_Q]$ is a finite free $\mathcal{O}[\Delta_Q]$-module.
- There should be at least 30 min left to explain the analogue of $(*)$: The proof is by induction on $|Q|$ and the theory of old-/new-forms.

References: lec. 25 from [C$^+$10]*, Taylor* [Tay06] *for some details*
Prerequisites: old/new automorphic forms, Hecke algebras: talks 1 and 9

Talk 14. 29.1. **The patching argument - Speaker: Konrad Fischer**
This talk presents the Taylor-Wiles-Kisin variant of techniques developed in [TW95]. At the heart of it lies an inverse-limit argument for modules over the family $\mathcal{O}[\Delta_{Q_n}]$. To construct an inverse system out of these data the pidgeon-hole principle makes a beautiful appearance.

- State our goal: The modularity lifting theorem (taking all reductions for granted).
- Recall properties of $\widetilde{R}^\square$ and $R_v, v \in St \cup S_p$ and outline how they transfer to properties (B1-B3) of $B := \widehat{\bigotimes}_{v \in St \cup S_p} R_v$ given in Prop. 2.1 of [C$^+$10], lec. 27.
- Define g, such that $B[\![x_1, \ldots, x_g]\!] \twoheadrightarrow R^\square$, $h := |Q|$ and j as number of framing variables. Recall the properties of $R_Q^\square$ for T-W-sets Q and show $h + j + 1 = \dim B + g$.
- Now recall the Hecke modules M_{Q_n} that we want to patch, frame them to get $M_{Q_n}^\square$ and prove compatibility (H1).
- First give a proof of an $R[\frac{1}{p}] \cong T[\frac{1}{p}]$ theorem assuming that we have an inverse system of rings and modules $R_{Q_n}^\square \circlearrowright M_{Q_n}^\square$ (section 3 of talk 27)
- Since there is a priori no relation between the different levels Q_n, to make this 'inverse limit' argument work explain the pidgeon-hole principle that makes the patching work: section 4 of *cite loc.*.

References: Follow lec. 27 in [C$^+$10]. *The last steps are also nicely explained in* [Gee13].
Prerequisites: talks 13 and 14.

Talk 15. 5.2. **Reserve date**

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