Oberseminar *Arithmetik von Zahl- und Funktionenkörpern* WS 2018

Shtuka cohomology and special values of Goss L-functions

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Introduction

Let K be a number field, $\mathcal{O}_K$ be its ring of integers. We set $K_\infty = \mathbf{R} \otimes_{\mathbf{Z}} K$ and consider the complex U_K of $\mathcal{O}_K$-modules given in degree 0 and 1 by

$$\left[\operatorname{Lie} \mathbf{G}_m(K_\infty) \longrightarrow \frac{\mathbf{G}_m(K_\infty)}{\mathbf{G}_m(\mathcal{O}_K)}\right], \tag{0.0.1}$$

induced by the exponential of the Lie group $\mathbf{G}_m(K_\infty)$ (mod $\mathbf{G}_m(\mathcal{O}_K)$). The degree 0 cohomology group of U_K is $\exp^{-1} \mathcal{O}_K^\times$ and Dirichlet's unit theorem yields that it is a finitely generated $\mathbf{Z}$-module. If ζ_K denotes the zeta function of K, the class number formula can be stated as

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \lambda \frac{\operatorname{covol}\left(H^0(U_K)\right)}{\operatorname{covol}(\mathcal{O}_K/\mathbf{Z})}$$

where λ is a non zero rational number and where both covolumes are calculated with respect to the same Haar mesure. Higher dimensional version of the class number formula are believed to be true, the Birch Swinnerton-Dyer conjecture is one example.

The aim of this seminar is to discuss the function field analogue of the latter where very recent progresses have been made. Following the work of Vincent Lafforgue and Lenny Taelman, Maxim Mornev just published his PhD thesis where he introduced a new cohomology theory for Drinfeld's shtukas.

Let C be a smooth projective curve over $\mathbf{F}_q$ and ∞ be a closed point on it. Let A be the ring of regular functions on C with at most a pole of finite order at ∞. We let F be the function field of C and denote K_∞ its completion at the infinite place. The usual analogy follows:

$$\mathbf{F}_q[t] \subseteq A \subset F \subset F_\infty \quad \sim \quad \mathbf{Z} \subseteq \mathcal{O}_K \subset K \subset \mathbf{R}. \tag{0.0.2}$$

Let R be a flat domain as a A-algebra and let K be its quotient field. For E is a Drinfeld module over R–the group scheme analogue of an elliptic curve in the function field setting–we define its Tate module at a prime $\mathfrak{p} \subset A$ as

$$T_{\mathfrak{p}} = \operatorname{Hom}_A\left(F_{\mathfrak{p}}/A_{\mathfrak{p}}, E(K^{sep})\right)$$

where K^{sep} is a fixed separable closure of K and $A_{\mathfrak{p}}$ (resp. $F_{\mathfrak{p}}$) is the completion of A (resp. F) at $\mathfrak{p}$. For $\mathfrak{m} \subset R$ distinct from the characteristic of E at $\mathfrak{p}$, the inverse characteristic polynomial $P_\mathfrak{m}$ of the geometric Frobenius at $\mathfrak{m}$ acting on $T_\mathfrak{p}$ has coefficients in A independent of the choice of $\mathfrak{p}$. In analogy with Artin's L-functions, we consider the infinite product:

$$L(E^*, 0) = \prod_{\mathfrak{m}} \frac{1}{P_\mathfrak{m}(1)}$$

that converges in F_∞ (which already breaks the analogy with $\mathbf{Z}$). Letting $K_\infty = F_\infty \otimes_A K$, Taelman has introduced the *unit complex* U_E which as the form very similar to (0.0.1):

$$U_E = \left[\operatorname{Lie} E(K_\infty) \longrightarrow \frac{E(K_\infty)}{E(R)} \right].$$

He proves that $H^0(U_E)$ is a finitely generated A-module and that $H^1(U_E)$ is finite, results which are highly reminiscent of Dirichlet's unit theorem. Computing the determinant of this complex of A-modules in the Knudsen-Mumford's sense and comparing it to the one of $\operatorname{Lie} E(R)$, we obtain Mornev's *class formula*:

Theorem 1 (Class formula)**.** Let E be a Drinfeld module over R with *everywhere good reduction.* As subsets of $\det_{F_\infty} \operatorname{Lie} E(K_\infty)$, we have

$$\det_A U_E = L(E^*, 0) \det_A \operatorname{Lie} E(R).$$

In particular, $\det_A U_E$ and $\det_A \operatorname{Lie} E(R)$ are isomorphic projective A-modules.

The main tool used in the proof is the cohomology of sthukas, the latter being objects in a bigger category than the Drinfeld modules. The finite flat A-algebra R is a Dedekind domain of finite type over $\mathbf{F}_q$. To such an algebra R one can functorially associate a smooth connected projective curve X over $\mathbf{F}_q$ together with an open embedding $\operatorname{Spec} R \subset X$. Let τ be the endomorphism of $\operatorname{Spec} A \times X$ given by the identity on $\operatorname{Spec} A$ and the q-power map on X.

Definition 2. A shtuka $\mathcal{M}$ on $\operatorname{Spec} A \times X$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

where $\mathcal{M}_0$ and $\mathcal{M}_1$ are coherent sheaves on $\operatorname{Spec} A \times X$ and

$$i : \mathcal{M}_0 \longrightarrow \mathcal{M}_1, \quad j : \mathcal{M}_0 \longrightarrow \tau^* \mathcal{M}_1,$$

are morphisms of coherent sheaves.

We will define study deeply various cohomology theories in the category of such objects. The main cohomology is defined as the derived cohomology of a particular Hom functor. As a result, if a shtuka $\mathcal{M}$ comes a Drinfeld module E, we have quasi-isomorphisms of complexes

$$\mathrm{R}\Gamma(\mathcal{M}) \cong U_E[-1], \tag{0.0.3}$$

$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \cong \operatorname{Lie} E(R)[-1]. \tag{0.0.4}$$

Given a shtuka $\mathcal{M}$, two type of maps will appear to be of fundamental importance for class formulas:

$$\zeta_\mathcal{M} : \det \mathrm{R}\Gamma(\mathcal{M}) \longrightarrow \det \mathrm{R}\Gamma(\nabla\mathcal{M}), \quad \rho_\mathcal{M} : F_\infty \otimes_A \mathrm{R}\Gamma(\mathcal{M}) \longrightarrow \mathrm{R}\Gamma(\nabla\mathcal{M}),$$

where $\nabla\mathcal{M}$ is a *linearlized version* of $\mathcal{M}$. If $\mathcal{M}$ is *ellitpic*, which is the case if it comes from a Drinfeld with everywhere good reduction, the $\zeta_{\mathcal{M}}$-map and the regulator $\rho_{\mathcal{M}}$ are linked by the formula

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \det_{F_\infty} \rho_{\mathcal{M}}. \tag{0.0.5}$$

If $\rfloor\mathcal{M}$ is a *shtuka model* for E, we have $L(\mathcal{M}) = L(E^*, 0)$. The combination of (0.0.3), (0.0.4) and (0.0.5) will give the class formula.

Talks

Talk 1 (Overview)**.** There are several categories occurring in the function field setting. This first talk should give an overview of the big picture and passages between those structures: Drinfeld $\mathbf{A}$-modules, $\mathbf{A}$-motives and Shtuka of $\mathcal{O}_X$-modules (everything should be contained in [4, Chapter 7] but one can be helped with [1, Chapter 4, 5 and 6]). The speaker will also link Drinfeld $\mathbf{A}$-modules to $\mathbf{A}$-lattices through the exponential map. It should be clear why Drinfeld modules are analogue of elliptic curves. Among other, the definition of the *exponential map*, the *field of definition*, *good reduction (everywhere)*, the *Tate module*, and special L-values have to appear.

In the remaining time, the speaker will define the complex of unit attached to a Drinfeld $\mathbf{A}$-module and state Mornev's *class number formula* (see [4, Introduction]). The example of the Carlitz module should be given and one should stress the analogy with $\mathbf{G}_m$ for the integer case.

Date: April 19, 2018 **Speaker: Quentin**

Talk 2 (Shtuka I)**.** The aim of this talk is to define Mornev's shtuka cohomology and define the *associated complex*. First of all, you will define the *mapping fiber* in Chapter Notations and Conventions. The speaker should then follow [4, Chapter 1, section 1-4] by recalling the definition of the category of shtukas (definition 1.2), explaining why it is abelian and why it has enough injective objects (theorem 2.6). The shtuka cohomology (section 4) will then be defined as the derived cohomology of the left-exact functor $\Gamma(X, -)$ of global sections (definition 4.2)[1]. Once the definition of the unit shtuka $\mathbb{1}_X$ given, the speaker will introduce the quasi-isomophism $\mathrm{RHom}(\mathbb{1}_X, -) \cong \mathrm{R}\Gamma(X, -)$ (theorem 4.6 through lemma 4.5).

You will then spend some time on the definition of the associated complex (definition 5.1). The talk should end with the exemple that follows.

Date: April 26, 2018 **Speaker: ?**

Talk 3 (Shtuka II)**.** This talk covers a second part of [4, Chapter 1] and contains the definition of a fundamental object: nilpotent shtukas. The speaker will start by properties of the associated complex, continuing the previous exposition of section 5. He will give an overview of the proof of theorem 5.6. To do so, the speaker might want to recall the definition of distinguished triangles and the associated complex from the previous talk. Some time should also been spent on how this complex behave through pull-back and push-forward with the help of section 6 and 7.

You will give theorem 8.1 on affine shtuka with section 8. After that, state the definition of a nilpotent shtuka together with proposition 9.3 and 9.4.

Date: May 3, 2018 **Speaker: ?**

[1] If needed, a crash course on derived left-exact functor cohomology is given in [2, Chapter 3, section I]

Talk 4 (Shtuka III)**.** This talk is devoted to the linearization functor and the ζ-morphism. To do so, one needs some (very brief) introduction on the theory of Knudsen-Mumford determinants in [3, Chapter 1]; it should be clear how to assigne to a perfect complex $\mathcal{F}^\bullet$ an invertible sheaf $\det \mathcal{F}^\bullet$ in such a way that det becomes functorial; also enlight the role of distinguished triangles in this theory. The lemma on determinant in [4, Chapter 6, section 6] should then be exposed with respect to the latter.

The remaining part will contain the definition of the ζ-morphism based on the determiant theory (the whole section 10). There should be a particular emphasis on the linearization functor ∇. The speaker will continue with the Hom shtuka exposed in section $11 - 12$ and the talk will end on theorem 12.5.

Date: May 10, 2018 **Speaker: ?**

Talk 5 (Elliptic shtukas)**.** The aim of this talk is to define *elliptic shtukas*, the main type of shtuka used to prove the class formula. First, the speaker will define the *completed tensor product* (its definition lies in the overview of Chapter 2). You will follow the definition by exemples in section 7, Chapter 3. Go to Chapter 5, and sketch the proof of lemma 3.1 and theorem 4.2. After defining elliptic shtukas in section 6, you will state and prove theorem 6.4. To end this talk, give an overview of section 7 on twists and quotients.

Date: May 17, 2018 **Speaker: ?**

Talk 6 (Regulators)**.** This talk deals with the regulator map, a crucial ingredient that one wants to link with the ζ-isomorphism. The speaker will, in a first part, explain as much as needed from *Artinian regulators* in Chapter 5, section 5. Then, state some results on the filtration on cohomology in section 8 of the same chapter.

In a second part, you will give the general definition of regulators in section 14. State theorem 14.4 and 14.5 without proofs, but you will outline the proof of 14.6.

Date: May 24, 2018 **Speaker: ?**

Talk 7 (Trace formula)**.** The aim of this talk is to state Mornev's Trace formula that links the regulator map to the ζ-isomorphism. It generalizes both Anderson and Lafforgue's results.

Date: May 31, 2018 **Speaker: ?**

Talk 8 (Survey of topological vector spaces)**.** This talk is an overview of chapters 2 and 3 on topological vector spaces, rings and modules over $\mathbf{F}_q$. It aims to give some feeling on the different completion processes and function spaces of interested for this thesis, without going deeply into details. This part is rather eavy in notations and it might be good to have an handout for the audience.

You will start his talk by covering the example in section 2 where $c(F/A, K)$ is described as a one-dimensional Tate algebra. Continue with the definition of the key words *topological vector space over* $\mathbf{F}_q$, *locally compact vector space* and *linearly topologized vector space* ([4, section 1]) together with the statement of lemmata $3.3 - 3.5$ and proposition 4.1. The continuous dual (definition 4.3) will be introduced to state an adaptation of Pontryagin duality with theorem 4.5.

The speaker should use section $5 - 7$ to state carefully the definition of the *completed*, and *ind-completed*, tensor product and the *hash* operation. The examples in [4, Chapter 3, section 7] should come right after that. You will define the topological spaces $c(V, W)$, $b(V, W)$, $a(V, W)$ and $g(V, W)$ (respectively section $8 - 11$). Some lights have to be shed on the links between those spaces. The completion process for topological algebra has also to appear (lemma [4, Chapter 3, 2.1]) and the speaker may return to examples in chapter 3, section 7 then. Definition 9.1 will be exposed.

The talk will finish on residue and duality with theorem 10.1 and some applications with the different corollaries in section 10 (if time allows).

Date: June 7, 2018 **Speaker: ?**

Talk 9 (Cohomology of Shtukas I)**.** The purpose of this talk is to introduce three cohomology theory for shtukas: the *germ*, *Čech* and *compactly supported cohomology*. You will first give some overview following [4, Chapter 4, introduction]. It is fundamental to keep track on which topological space we are working for each cohomology. A important assumption to compare those cohomology is the nilpotence of $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$. This assumption as to be clear as possible.

For the germ cohomology, it seems to be enough to state the definition 1.1 together with proposition 2.1. The definition of the *local germ map* also need to appear.

Next, you will examine the Čech cohomology. Briefly outline section 4; make as clear as possible the quasi-isomorphism of complexes: $\mathrm{R}\check{\Gamma}(S \times X, \mathcal{M}) \cong \mathrm{R}\Gamma(S \times X, \mathcal{M})$, under the nilpotence assumption (theorem 4.11).

The last part of this talk should contain the main definitions on compactly supported cohomology, as definition 5.1 and proposition 5.2 on $\mathrm{R}\Gamma_c(S \otimes R, \mathcal{M}) \cong \mathrm{R}\Gamma(S \times X, \mathcal{M})$, under the nilpotence assumption. You will end on the definition (5.3) of the *global germ map*.

Date: June 14, 2018 **Speaker: ?**

Talk 10 (Cohomology of Shtukas II)**.** This talk will examine the remaining parts of [4, Chapter 4]. As in the previous talk, you will keep a carefull track of the space you are working on. At first, recall the previous definitions of the local and global germ maps (definition 2.4 and 5.3), and follow mostly section 6 discussing briefly theorem 6.1. You will then reproduce as much as possible the material of section 7 on completed Čech cohomology. Theorem 7.10 and 7.11 have to be – at least – stated.

In a second part, the speaker should cover sections 8 and 9 on the *change of coefficients* and how the ζ-isomorphism behave under it. Attention has to be mostly focus on proposition 9.1 and 9.2.

Date: June 21, 2018 **Speaker: ?**

Talk 11 (Regulators I)**.** This talk is the first of a series of three on *regulators* in Chapter 5. The purpose of it is to assimilate the nilpotence property of shtukas with properties of their cohomology groups, and state finitness results on the latter. An important part of it is also devoted to *Artinian regulators*.

You will state lemma 3.1 and outline its proof. Theorem 4.2 should also be exposed. You can help yourself with sections 1 and 2 of Chapter 5 without spending too much time on them. The second part is devoted to Artinian regulators in section 5. For the exposition, state carefully definition 5.2 and give as much material as possible from section 5.

Date: June 28, 2018 **Speaker: ?**

Talk 12 (Regulators II)**.** The main topic of this second talk on Chapter 5 are *elliptic shtukas*. It is also to define general *regulators*. The talk contains some technicalities involving twisting, change of coefficients and filtration on the cohomology for elliptic shtukas. It will end on a general definition of *regulators* in this flavour.

The definition of elliptic shtukas in section 6 will be stated carefully, recalling some background from the previous talks when needed. Some time should be spent on the example of the Carlitz module. The technicalities of section 7 and 8 will be outlined at the convinience of the speaker but Proposition 7.6 has to appear.

Finish the talk on the overview of regulators reproducing the material in section 9.
Date: July 5, 2018 **Speaker: ?**

Talk 13 (Regulators III)**.** TO BE CONTINUED **Date: July 12, 2018** **Speaker: ?**

Talk 14 (Trace formula I)**. Date: July 19, 2018** **Speaker: ?**

Talk 15 (Trace formula II)**. Date: July 26, 2018** **Speaker: ?**

References

[1] D. Goss (1991); *Basic structures of function fields*, Springer.

[2] R. Hartshorne (1977); *Algberaic Geometry*, Springer.

[3] F. Kundsen, D. Mumford (1977); *The Projectivity of the Moduli Space of Stable Curves. I: Preliminaries on* det *and Div*, Mathematica Scandinavica, vol. 39, no. 1, p. 15–55 (available at `http://www.mscand.dk/article/view/11642/0`).

[4] M. Mornev (2018); *Shtuka cohomology and special values of Goss L functions* (available at `https://people.math.ethz.ch/~mmornev/`).

[5] L. Taelman (2008); *Special L-values of t-motives: a conjecture* (preprint version available at `https://arxiv.org/abs/0811.4522`).