p-divisible Groups **April 2019**
Dr. J. Ludwig

1. Exercise Sheet

Exercise 1:
Let R be a ring and let F be the functor that associates to each R-algebra A the set of its idempotent elements.

1. Show that the functor F is represented by the R-algebra $R[X]/(X^2 - X)$.
2. Show that the maps $F(A) \times F(A) \to F(A)$ given by $(e, e') \mapsto e + e' - 2ee'$ induce natural group structures on the sets $F(A)$.
3. Show that $G = \mathrm{Spec}(R[X]/(X^2 - X))$ has a group scheme structure that induces the group laws of part (2).
4. Prove that G is isomorphic to the constant group scheme $\underline{\mathbb{Z}/2\mathbb{Z}}_R$.

Exercise 2:
Let k be a field of characteristic $p > 0$ and let $W(X, Y)$ denote the polynomial $((X + Y)^p - X^p - Y^p)/p \in \mathbb{Z}[X, Y]$.

1. Show that the k-scheme $\mathrm{Spec}(k[X, Y]/(X^p, Y^p))$ with group law given by
$$(x, y) + (x', y') = (x + x', y + y' - W(x, x'))$$
is a group scheme.
2. Compute the Cartier dual of α_{p^2}; show it is isomorphic to the group scheme of part (1). Here α_{p^2} denotes the closed subgroup scheme of $\mathbb{G}_a$ given by $\alpha_{p^2}(A) = \{x \in A : x^{p^2} = 0\}$ for any k-algebra A.

Exercise 3:
Let R be a ring and let $a, b \in R$ such that $ab = 2$. Put $A := R[X]/(X^2 + aX)$.

1. Show that $\mu : A \to A \otimes_R A$, $\mu(X) = X \otimes 1 + 1 \otimes X + bX \otimes X$ defines an R-algebra homomorphism, which together with $\varepsilon : A \to R, \varepsilon(f) = f(0)$, and $\iota = id : A \to A$ turns A into a Hopf algebra. The resulting group scheme is denoted by $G_{(a,b),R} = \mathrm{Spec}(A)$.
2. Show that $G_{(a,b),R} \cong G_{(a',b'),R}$ if and only if there is a unit $u \in A^\times$ with $a = ua'$ and $b = 1/ub'$.
3. Show that the Cartier dual of $G_{(a,b),R}$ is given by $G_{(b,a),R}$.

Remark: One can show that if $G = \mathrm{Spec}(A)$ is an affine group scheme over R, with A free of rank 2 over R, then in fact $G \cong G_{(a,b),R}$ for some a, b as above.