p-divisible Groups **May 2019**
Dr. J. Ludwig

2. Exercise Sheet

Exercise 1:

Let R be a commutative ring.
1. Let $A = R[x, y]/(xy)$. Compute the relative differentials $\Omega^1_{A/R}$.
2. Let $A = R[x_1, \dots x_n]/(f_1, \dots f_m)$. Show that $\Omega^1_{A/R}$ can be interpreted as the cokernel of the Jacobian matrix $\mathcal{J}$

$$\mathcal{J} = (\partial f_j/\partial x_i) : A^m \to A^n.$$

3. Let $A = \mathbb{Q}[[x_1, \dots, x_n]]$ be the ring of formal power series in $n \geq 1$ variables. Prove that $\Omega^1_{A/\mathbb{Q}}$ is not finitely generated.

Exercise 2: Show that any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, show that there exists a presentation $S = R[x_1, \dots, x_n]/(f_1, \dots, f_n)$ such that the image of $\det(\partial f_j/\partial x_i)$ is invertible in S.

Exercise 3:

Let R be a complete local Noetherian ring. Let H be a connected finite flat group scheme over $\mathrm{Spec}(R)$ and G a finite étale group scheme over $\mathrm{Spec}(R)$. Show that any map $H \to G$ is the zero map.

Aufgabe 4:

Let k be a non-perfect field and let $k' := k(u^{1/p})$ be an inseparable field extension. For $i = 0, \dots, p-1$, let

$$A_i = k[t]/(t^p - u^i)$$

and set

$$A := \prod A_i.$$

Let $G^0 := \mathrm{Spec}(A)$ and $G_i := \mathrm{Spec}(A_i)$ and define a multiplication by

$$G_i(R) \times G_j(R) \to G_{i+j}(R), (f_i, f_j) \mapsto (f_{i+j} : t \mapsto f_i(t)f_j(t)), \text{ for } i + j \leq p - 1,$$

$$G_i(R) \times G_j(R) \to G_{i+j-p}(R), (f_i, f_j) \mapsto (f_{i+j} : t \mapsto f_i(t)f_j(t)/u), \text{ for } i + j \geq p,$$

where R is any k-algebra.
Show that this turns G into a finite flat group scheme with $G^0 = \mu_{p,k}$. Show that the connected-étale sequence of G is given by

$$0 \to \mu_{p,k} \to G \to \underline{\mathbb{Z}/p\mathbb{Z}}_k \to 0.$$

Show that this sequence does not split.