Exercises on adic spaces

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This is a selection of exercises on adic spaces, that I recently used as I was assisting a course on the subject[1]. The exercises are grouped according to topics. Most exercises are taken from examples and remarks in the lecture notes of Wedhorn [W] and of Morel [Mo], some are inspired by notes from Conrad's number theory learning seminar on perfectoid spaces [C] or taken from a course by Görtz [G] and there are some new ones as well, e.g. on specializations.[2] All exercises are supposed to help you understand the theory of adic spaces when first learning it (either in a course, or using [W] or [Mo]).[3]

Notation: A non-archimedean field K is a field that is complete with respect to a non-archimedean valuation of rank 1. We use the terminology and notations from [W], with the exception that we use the phrase Huber ring (instead of f-adic ring) and Huber pairs (instead of affinoid rings).

Totally ordered abelian groups

Exercise 1. Let Γ be a totally ordered abelian group with $\Gamma \neq 1$.

(a) Show: For each $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ with $\gamma' < \gamma$.

(b) Let $\Delta \subset \Gamma$ be a convex subgroup. Show that

$$\mathrm{ht}(\Gamma) = \mathrm{ht}(\Delta) + \mathrm{ht}(\Gamma/\Delta).$$

Exercise 2. Show that the group $\mathbb{Z} \times \mathbb{Z}$ can be equipped with uncountably many different total orders of height 1.

Exercise 3. Show that for a totally ordered abelian group $\Gamma \neq \{1\}$ the following are equivalent:

(a) $\mathrm{ht}(\Gamma) = 1$.

(b) There is an injective homomorphism $\Gamma \hookrightarrow \mathbb{R}_{>0}$ of totally ordered groups.

(c) Γ is archimedean, i.e. for all $\gamma, \delta \in \Gamma_{<1}$ there exists $m \in \mathbb{N}$ with $\delta^m < \gamma$.

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[1] https://typo.iwr.uni-heidelberg.de/groups/arith-geom/home/members/judith-ludwig/adischeraeume1/.The goal of the course was to develop the foundations of the theory of adic spaces.

[2] This list of exercises is a start to make up for the lack of a textbook on adic spaces. I thought it would be a good idea to collect helpful exercises. All errors are mine. This document is supposed to expand in time, if you have suggestions for exercises or corrections, please send them to judith.ludwig at iwr.uni-heidelberg.de

[3] For further exercises, see the nice collection [Mie] from Mieda for the AWS 2017 with very little overlap with this list.

Valuations and valuation rings

Exercise 4. Let $(K, |\cdot|)$ be a non-archimedean field. Consider the group $\Gamma = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ with lexicographic order, i.e.

$$(a,b) < (a',b') :\Leftrightarrow a < a' \text{ or } (a = a' \text{ and } b < b').$$

Then $\Gamma \cup \{0\}$ gets the structure of a totally ordered monoid. Let $\gamma = (1, \gamma_0) \in \Gamma$ with $\gamma_0 < 1$. For $r \in \mathbb{R}_{>0}$ we write r for $(r, 1)$. Show:

(a) For all $0 < r < 1$, we have $r < \gamma < 1$.

(b) The map

$$v : K\langle T \rangle \to \Gamma \cup \{0\}, \ \sum_{n \geq 0} a_n T^n \mapsto \max_{n \geq 0} |a_n| \gamma^n.$$

defines a valuation.

Exercise 5. Let R be a valuation ring with fraction field $K := \mathrm{Quot}(R)$. Show:

(a) R is integrally closed in K.

(b) Each finitely generated ideal in R is a principal ideal.

(c) An R-module M is flat if and only if it is torsion free.

Exercise 6. (i) Let K be a field. Let $B \subset A \subset K$ be two subrings, with B a valuation ring of K. Show:

(a) A is also a valuation ring of K.
(b) $\mathfrak{m}_A \subset \mathfrak{m}_B$, with equality if and only if $A = B$.
(c) $\mathfrak{m}_A$ is a prime ideal of B and $A = B_{\mathfrak{m}_A}$.
(d) $B/\mathfrak{m}_A$ is a valuation ring of $A/\mathfrak{m}_A$.

(ii) Let $A \subset K$ be a valuation ring. Let $\overline{B}$ be a valuation ring of $A/\mathfrak{m}_A$ and let B be the preimage (under the canonical projection $\pi \colon A \to A/\mathfrak{m}_A$) of $\overline{B}$ in A. Show:

(a) B is a valuation ring of K (in particular $K = \mathrm{Quot}(B)$).
(b) We have $\mathfrak{m}_B = \pi^{-1}(\mathfrak{m}_{\overline{B}})$.

(iii) We can illustrate these assertions with the following example. Let k be a field and define $K = k((u))((t))$. Consider the valuation ring $A = k((u))[[t]]$ of the t-adic valuation on K. Then $A/\mathfrak{m}_A = k((u))$ has the u-adic valuation with valuation ring $\overline{B} = k[[u]]$. Determine $B = \pi^{-1}(\overline{B})$, the maximal ideal $\mathfrak{m}_B$, the rank and the value group of B.

Exercise 7. Let K be a field and $R \subset K$ be a valuation ring. Show that the rank of R is equal to the Krull dimension of R.
Hint: Use [W, Prop. 2.14].

Exercise 8. (Example of a non-henselian valuation ring[4])
Let K be a field and consider the ring of formal power series $K[[Y,Z]]$. Let $L := \mathrm{Quot}(K[[Y,Z]])$ be the fraction field. Define a valuation $|\cdot|\colon L \to \gamma^{\mathbb{Z}} \cup \{0\}$ of L by $|Y^i| = \gamma^{-i}$ for all $i \in \mathbb{Z}$ and $|f(Z)| = 1$ for all $f(Z) \in K[[Z]]$.

(a) Verify that $|\cdot|$ does indeed define a valuation.

(b) Show that the valuation ring $R := \left\{\frac{f}{g} \in L : |f| \leq |g|\right\}$ is not henselian:
For that consider the polynomial $P(T) = T^2 - T + \frac{Y}{Z} \in R[T]$. Let $\mathfrak{m} \subset R$ be the maximal ideal. Then the reduction $\overline{P}(T) = T(T-1)$ modulo $\mathfrak{m}$ has two roots in $R/\mathfrak{m}$. Show that $P(T)$ has no root in R.

Exercise 9. Let K be a field, $|\cdot|$ a valuation on K with value group Γ. Let Γ' be a totally ordered abelian group, with $\Gamma \subset \Gamma'$ and let $\gamma \in \Gamma'$. Show:

(a) There is an extension $|\cdot|'$ of $|\cdot|$ to $K(T)$ with values in Γ' and such that $|T|' = \gamma$.

(b) If the image of γ in Γ'/Γ has infinite order, then there exists a unique valuation
$$|\cdot|' : K(T) \to \Gamma',$$
that extends $|\cdot|$ and has the property that $|T|' = \gamma$. We have an isomorphism of residue fields $\kappa(|\cdot|') \cong \kappa(|\cdot|)$ and $\Gamma_{|\cdot|'} = \Gamma \cdot \gamma^{\mathbb{Z}}$.

Spectral spaces and the constructible topology

Exercise 10. Show the following assertions:

(a) Any locally spectral space is sober.

(b) Each locally closed subspace of a sober space is sober.

(c) Each quasi-compact open subspace of a spectral space is spectral.

(d) Each locally spectral space has a basis of open spectral subspaces.

Exercise 11. Let $X := \mathbb{N} \cup \{\infty\}$ with the usual total order and the topology with open sets $\{X_{<x} : x \in X\} \cup \{X\}$. Show:

(a) X is a spectral space.

(b) $X \cong \mathrm{Spec}(A)$ for a valuation ring A with value group $\prod_{\mathbb{N}} \mathbb{Z}$ (w. lexikographic order).

Let Y be defined by glueing two copies of X along $X_{<\infty}$. Show that Y is locally spectral and quasi-compact, but not quasi-separated. In particular Y is not locally spectral.

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[4] Thanks to A. Maurischat for suggesting this exercise.

Exercise 12. Let X be a spectral space. Show that the constructible topology is the coarsest topology, such that all closed subspaces of X and all quasi-compact subspaces of X are closed.

Exercise 13. Let X be a finite Kolmogorov space. Show:

(a) X is spectral.

(b) Each subset of X is constructible.

Exercise 14. Let X be a spectral space and Z be a subspace. Show:

(a) Z is pro-constructible if and only if Z is spectral and the inclusion $Z \hookrightarrow X$ is spectral.

(b) For a pro-constructible subset Z of X, the closure of Z coincides with the set of specializations of points in Z.

Riemann–Zariski spaces

Exercise 15. *Let K be a field, $A \subset K$ a subring. The* Riemann–Zariski space *of K over A is the set*

$$RZ(K,A) := \{R \supset A : \ R \text{ is a valuation ring of } K\}.$$

On $RZ(K,A)$ we install the topology, with a basis of open subsets given by

$$U(x_1,\dots,x_n) = \{R \in RZ(K,A) : x_1,\dots,x_n \in R\}, \ \text{for } x_1,\dots,x_n \in K.$$

If A is the image of $\mathbb{Z}$, we write $RZ(K,\mathbb{Z}) = RZ(K)$ and call it the Riemann–Zariski space *of K.*
Show:

(a) There is a homeomorphism $RZ(K) \cong \mathrm{Spv}(K)$.

(b) Let $R, R' \in RZ(K,A)$. Then R is a specialization of R' in $RZ(K,A)$ if and only if $R \subset R'$.

(c) For a ring B and $\mathfrak{p} \in \mathrm{Spec}(B)$ the fibre $supp^{-1}(\mathfrak{p})$ of the support map $supp : \mathrm{Spv}(B) \to \mathrm{Spec}(B)$ is homeomorphic to $RZ(\mathrm{Quot}(B/\mathfrak{p}))$.
Remark: In particular with part (b) we see that for two elements $v, w \in \mathrm{Spv}(B)$ with the same support $\mathfrak{p}$, $v \in \overline{\{w\}}$ if and only if $R_v \subset R_w \subset \mathrm{Quot}(B/\mathfrak{p})$.

Exercise 16. Go on a quest to find out why there is a homeomorphism $RZ(\mathbb{C}(T),\mathbb{C}) \cong \mathbb{P}^1_{\mathbb{C}}$.

Specializations

Exercise 17. For a non-archimedean field K consider the valuation spectrum $\mathrm{Spv}(K\langle T\rangle)$ of the Tate algebra in one variable. Show, that the valuation v of exercise 4 is a specialization of the Gaußnorm $|\cdot|$, i.e. that $v \in \overline{\{|\cdot|\}}$.

Exercise 18. Show:

(a) Let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$. Then $\mathfrak{p}$ is a specialization of $\mathfrak{p}'$ if and only if $\mathfrak{p} \supset \mathfrak{p}'$.

(b) Let $v, w, w' \in \mathrm{Spv}(A)$. If w and w' are two vertical generalizations of v, then either w is a generalization of w' or w' is a generalization of w.

(c) Let $v, w \in \mathrm{Spv}(A)$ and let w be a horizontal specialization of v. Let $v' \in \mathrm{Spv}(A)$ be a vertical specialization of v. Show: There exists a unique vertical specialization w' of w, such that w' is a horizontal specialization of v'. The following diagram is a visualisation of this situation.

$$\begin{array}{ccc} v & \longmapsto & w \\ \downarrow & & \vdots \\ v' & \dashrightarrow & \exists! w' . \end{array}$$

Exercise 19. *For B an integral domain, we denote by $v_{B,triv}$ the trivial valuation, i.e.*

$$v_{B,\mathrm{triv}} \colon B \to \{0,1\}, v_{B,\mathrm{triv}}(f) = 1, \ \forall f \neq 0, v_{B,\mathrm{triv}}(0) = 0.$$

Let K be a field.

(a) Let $A = K[[X]]$ with valuation spectrum $\mathrm{Spv}(A)$. Consider the valuations $v_{A,\mathrm{triv}}$ and $w := v_{K,\mathrm{triv}} \circ p_x$, where $p_x : A \to K[[X]]/(X) \cong K$ is the canonical projection. Realize w as a horizontal specialization of a vertical specialization of $v_{A,\mathrm{triv}}$.

(b) Let $A = K[[X,Y]]$. Show:

  (a) The trivial valuation $v_{\mathrm{triv}} := v_{A,\mathrm{triv}}$ is a generic point of $\mathrm{Spv}(A)$.

  (b) Consider the following valuations on A

   - Let $\Gamma_v := \gamma^{\mathbb{Z}} \times \tau^{\mathbb{Z}}$ be equipped with the lexicographic order. Define
   $$v : A \to \Gamma_v \cup \{0\}, \ \sum a_{(i,j)} X^i Y^j \mapsto (\gamma^{-i_0}, \tau^{-j_0}),$$
   with $(i_0, j_0) = \inf\{(i,j) \in \mathbb{N}_0^2 : a_{(i,j)} \neq 0\}$.
   - Let $v_x \colon A \to \gamma^{\mathbb{Z}} \cup \{0\}$ be defined by $|X^i| = \gamma^{-i}$ for all $i \in \mathbb{Z}$ and $|f(Y)| = 1$ for all $f(Y) \in K[[Y]]$.
   - $w_{y,\mathrm{triv}} = v_{K[[Y]],\mathrm{triv}} \circ p_y$, where $p_y : A \to K[[X,Y]]/(X) \cong K[[Y]]$ is the canonical projection.

  Show: $w_{y,\mathrm{triv}}$ is a horizontal specialization of v_x and v is a vertical specialization of v_x. Complete the following diagram:

$$\begin{array}{ccc} v_x & \longmapsto & w_{y,\mathrm{triv}} \\ \downarrow & & \vdots \\ v & \dashrightarrow & \exists! w . \end{array}$$

Exercise 20. Consider $\mathrm{Spv}(\mathbb{Q}_p[T])$. Let $|\cdot|_p$ denote the p-adic absolute value on $\overline{\mathbb{Q}}_p$.

(a) For $a \in \overline{\mathbb{Q}}_p$ define a valuation by

$$|\cdot|_a : \mathbb{Q}_p[T] \to \mathbb{R}_{>0} \cup \{0\},\ f \mapsto |f(a)|_p.$$

Show: The point $|\cdot|_a$ is closed in $\mathrm{Spv}(\mathbb{Q}_p[T])$.

(b) For each $r \in \mathbb{R}_{>0}$ define a valuation $|\cdot|_r$ auf $\mathbb{Q}_p[T]$ as follows: For $f = \sum a_n T^n \in \mathbb{Q}_p[T]$, let

$$|f|_r = \max_{n\geq 0}\{\ |a_n|_p r^n\ \} = \sup_{x \in \overline{\mathbb{Q}}_p, |x|_p \leq r} |f(x)|_p.$$

Show that the valuation $|\cdot|_r$ has no proper horizontal specializations. Does $|\cdot|_r$ have vertical specializations?

Exercise 21. Let K be a field, $v \in \mathrm{Spv}(K)$ a non-trivial valuation with valuation ring $R \subset K$. Show that the following statements are equivalent.

(a) There is a valuation w of rank 1, such that w generates the same topology as v on K.

(b) There is a topologically nilpotent non-zero element in K.

(c) R has a prime ideal of height 1.

We say a valuation v on a field is microbial, if it satisfies these conditions.
Show that if v has finite rank, then v is microbial.
Hint for the proof of the equivalences: Show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).
For (b) $\Rightarrow$ (c): Show first of all, that K is a Huber ring with pair of definition $(R, \varpi R)$, for a suitable topologically nilpotent element ϖ. Now let $\mathfrak{p}$ be a prime ideal, which is minimal with the property that $I \subset \mathfrak{p}$. Show, that such a $\mathfrak{p}$ has height 1.

Adic morphisms of Huber rings

Exercise 22. Show that adic morphisms are continuous.

Exercise 23. Let A and B be Huber rings and $f : A \to B$ a ringhomomorphism.

(a) Show the converse to the previous exercise is wrong by giving an example of a continuous morphism from $A = \mathbb{Z}_p$ to $B = \mathbb{Z}_p[[T]]$ which is not adic.

(b) Show: If f is adic then f is bounded (i.e. for each bounded subset $T \subset A$, $f(T)$ is bounded.)

(c) Show that continuous morphisms of Huber rings are not necessarily bounded.

Remark: Adic morphisms are a good class of continuous morphisms, that preserve boundedness.

Exercise 24. Let $f : A \to B$ be a continuous morphism of Huber rings. Show that if A is Tate, then so is B and the morphism f is adic.

Exercise 25. (*Tensorproducts of Huber rings*)

(a) Let A, B and C be Huber rings. Let $f : A \to B$ and $g : A \to C$ be *adic* morphisms of Huber rings. Let (A_0, I) be a pair of definition for A, and let B_0 and C_0 be rings of definition of B and C, such that $f(A_0) \subset B_0$ and $g(A_0) \subset C_0$. Let D_0 be the image of $B_0 \otimes_{A_0} C_0$ in $D := B \otimes_A C$ and let J be the ideal of D_0 that is generated by the image $f(I)$.
Show: D is a Huber ring with pair of definition (D_0, J). The natural ring homomorphisms $u : B \to D$ and $v : C \to D$ are adic. (*)

Show furthermore: For each non-archimedean ring D' and each pair of continuous ringhomomorphisms $(u' : B \to D', v' : C \to D')$ with $u' \circ f = v' \circ g$ there is a unique continuous ringhomomorphism $\varphi : D \to D'$ such that $u' = \varphi \circ u$ and $v' = \varphi \circ v'$. If D' is a Huber ring, and u' and v' are adic morphisms, then φ is adic as well.

(b) The following example shows the importance of the requirement that f and g should be adic.[5]:
Let $A = A_0 = \mathbb{Z}_p$ be equipped with the p-adic topology. Let $B = B_0 = \mathbb{Z}_p[[x]]$ be equipped with the (p, x)-adic topology and $C = \mathbb{Q}_p \supset C_0 = \mathbb{Z}_p$ with the p-adic topology. Then the natural map $A \to C$ is adic, but the natural map $A \to B$ is not adic. We have $D = \mathbb{Z}_p[[x]][p^{-1}]$ and $D_0 = \mathbb{Z}_p[[x]]$. Show: There is no ideal $J \subset D_0$, which satisfies (*).

Remark: Using exercise 24 we see that the tensor product of Tate rings is a Tate ring.

Exercise 26. Let A be a Huber ring and $T = \{T_1, \dots, T_n\}$ where for $i = 1, \dots, n$, $T_i \subset A$ is a finite non-empty subset, with $T_i U$ open for each additive subgroup $U \subset A$.
Show: The non-archimedean A-algebra $A[X]_T$ is initial among all non-archimedean A-algebras $f : A \to B$ with ordered n–tuples $(b_1, \dots, b_n)$ with $b_i \in B$, such that the elements $f(t_{ij}) b_i \in B$ are power bounded for all i and all $t_{ij} \in T_i$.

Exercise 27. Let $A = A_0 = \mathbb{Z}_p[[x]]$ be equipped with the (p, x)-adic topology. This is an adic ring.

(a) Let $T := T_1 = \{1\}$ and $s = p \in A$. Calculate $A\langle T/s \rangle$.

(b) Let $T := T_1 = \{p, x\}$ and $s_1 = p, s_2 = x$.
Determine rings of definitions B_1 resp. B_2 for $A(T/s_1)$ resp. $A(T/s_2)$ and show, that $s_1 \notin B_1^\times$ and that $s_2 \notin B_2^\times$. Show, that the topology on B_1 agrees with the p-adic topology.
Show furthermore, that $A\langle T/s_1 \rangle$ and $A\langle T/s_2 \rangle$ are not adic.

[5]…and indicates, that one has to be cautious when constructing fibre products of adic spaces.

The spectrum of continuous valuations

Exercise 28. Let K be a field and let v be a microbial valuation on K. We equip K with the valuation topology induced from v. Show:
$\mathrm{Cont}(K) \subset \mathrm{Spv}(K)$ consists of those valuations w, whose valuation topology on K coincides with that of v. Furthermore, show that $\mathrm{Cont}(K) = \mathrm{Cont}(K)_{an}$.

Exercise 29. Let $A = \mathbb{Z}_p[[x]]$ be equipped with the (p,x)-adic topology. Determine $\mathrm{Cont}(A) \backslash \mathrm{Cont}(A)_{an}$, i.e. all non-analytic continuous valuations on A.

Exercise 30. Let A be a Huber ring. Show that the natural map

$$\mathrm{Cont}(\hat{A}) \to \mathrm{Cont}(A)$$

is bijective.
Hint: An adic ring (with ideal of definition I) is Hausdorff if and only if $\bigcap_n I^n = \{0\}$.

Huber pairs and adic spectra

Exercise 31. Let K be a non-archimedean field. Consider the subring

$$B := \left\{ \sum_{i \geq 0} a_i T^i \in K^\circ\langle T \rangle : \ |a_i| < 1, \forall i \geq 1 \right\} \subset K^\circ\langle T \rangle.$$

Show that $(K\langle T \rangle, K^\circ\langle T \rangle)$ and $(K\langle T \rangle, B)$ are Huber pairs.

Exercise 32. Let $h\colon (B,B^+) \to (C,C^+)$ be an adic morphism of Huber pairs and $\mathrm{Spa}(h)\colon Y = \mathrm{Spa}(C,C^+) \to X = \mathrm{Spa}(B,B^+)$ the induced map between the adic spectra. Show:

(a) We have
$$\mathrm{Spa}(h)^{-1}\left(X\left(\frac{T}{s}\right)\right) = Y\left(\frac{h(T)}{h(s)}\right),$$
i.e. the preimage of a rational subset is rational.

(b) $\mathrm{Spa}(h)$ is spectral.

Exercise 33. Let (A,A^+) be a Huber pair. Show that $\mathrm{Spa}(A,A^+) \subset \mathrm{Cont}(A)$ is dense and that it contains all trivial valuations of $\mathrm{Cont}(A)$ and all rank 1 points of $\mathrm{Cont}(A)_{an}$.
Hint: Show, that any $x \in \mathrm{Cont}(A)$ is a vertical specialization of an element $y \in \mathrm{Spa}(A,A^+)$.

Exercise 34. Let K be a non-archimedean field. Find an element $v \in \mathrm{Cont}(K\langle T\rangle)$ that is not contained in $\mathrm{Spa}(K\langle T\rangle, K^\circ\langle T\rangle)$.
Hint: Modify the valuation from exercise 4.

Exercise 35. Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid adic space and let $W = \mathrm{Spa}(B, B^+)$ be an affinoid open subspace of X. Let $x \in W$. Show that x is analytic as a valuation on A if and only if x is analytic as a valuation on B.
Hint: Reduce to the case of W being a rational subset of X.

Exercise 36. For this exercise you may use the following important fact (see [Mo, Theorem III.3.1]):
Let B be a complete Huber ring and let $f_1, \dots, f_n, g \in B$ be elements, such that $f_1, \dots, f_n$ generate an open ideal in B. Then there is a neighbourhood V of 0 in B, such that for all $f_1', \dots, f_n', g' \in B$: If $f_i' \in f_i + V$ for $i \in \{1, \dots, n\}$ and $g' \in g + V$, then the ideal of B generated by $f_1', \dots, f_n'$ is open and

$$\mathrm{Cont}(B)\left(\frac{f_1, \dots, f_n}{g}\right) = \mathrm{Cont}(B)\left(\frac{f_1', \dots, f_n'}{g'}\right).$$

Now let (A, A^+) be a Huber pair. Then its completion $(\hat{A}, \hat{A}^+)$, where $\hat{A}^+$ denotes the closure of A^+ in A, is again a Huber pair.
Show that the natural map $h : \mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$ preserves rational subsets, i.e. a subset of $\mathrm{Spa}(\hat{A}, \hat{A}^+)$ is rational if and only if its image in $\mathrm{Spa}(A, A^+)$ is rational.

Exercise 37. (Example of a non-sheafy Huber pair)
Let K be a non-archimedean field with valuation ring $\mathcal{O}$. Let $\varpi \in \mathcal{O}$ be an element with $0 < |\varpi| < 1$. Let $R = K[T, T^{-1}, Z]/(Z^2)$. Let R_0 be the $\mathcal{O}$–submodule R with $\mathcal{O}$-basis

$$\varpi^{|n|}T^n,\ n \in \mathbb{Z}, \varpi^{-|n|}T^n Z,\ n \in \mathbb{Z}.$$

(Here $|\cdot|$ denotes the archimedean absolute value on $\mathbb{Z}$).

(a) Verify, that R_0 is an $\mathcal{O}$-subalgebra of R with $KR_0 = R$. Show that the ideal $ZR \cap R_0 \subset R_0$ is not finitely generated, so that in particular R_0 is not noetherian.

Equip R with the topology making $\varpi^N R_0$ a fundamental system of open neighbourhoods of 0. Consider the Huber pair (R, R°). Let $X = \mathrm{Spa}(R, R^\circ)$. The goal now is to show that the presheaf $\mathcal{O}_X$ is not a sheaf. For that let

$$U = \{x \in X : |T(x)| \le 1\},$$

$$V = \{x \in X : |T(x)| \ge 1\}.$$

(b) Show: U and V are rational domains of X and $U \cup V = X$.

(c) Show, $Z \neq 0$ in $\mathcal{O}_X(X)$, but $Z = 0$ in $\mathcal{O}_X(U)$ and in $\mathcal{O}_X(V)$. In particular, the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective.

This example shows that even for good Huber rings (R is Tate, and finitely generated as a K-algebra) the sheaf properties can be violated by choosing a nasty ring of definition. This and more examples of non-sheafy Huber pairs can be found in [BV].

Exercise 38. (The affine line over $\mathbb{Q}_p$.)
For $n \in \mathbb{N}_0$, let $D_n = \operatorname{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ be the closed adic unit disc over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Consider the morphisms

$$\varphi_n : D_n \to D_{n+1}$$

induced by the ring map $\mathbb{Q}_p\langle T\rangle \to \mathbb{Q}_p\langle T\rangle, T \mapsto pT$. Show:

(a) The direct limit

$$\mathbb{A}^1_{\mathbb{Q}_p} := \varinjlim_{\varphi_n} D_n,$$

is an adic space which is not quasi-compact.

(b) $D = D_0 \subset \mathbb{A}^1_{\mathbb{Q}_p}$ is not closed.

(c) Determine the closure of D_0 in $\mathbb{A}^1_{\mathbb{Q}_p}$.

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