Summary of lectures on p-divisible groups (2019)

finite flat group schemes G (ffgs)

étale group schemes

Cartier dual G^*, $G^* \cong \mathrm{Hom}(G, \mathbb{G}_{m,R})$

constant group schemes Γ_R, (Γ finite abelian gp)

examples: $\alpha_{p,R}$, $\mu_{p^n,R}$, $\mathbb{Z}/l^n\mathbb{Z}_R$

special kind of

Affine group schemes

$G = \mathrm{Spec}(A)/\mathrm{Spec}(R)$

affine subgroup schemes

maps $f : G \to H$

Kernel $ker(f)$

examples: $\mathbb{G}_{a,R}$, $\mathbb{G}_{m,R}$

Translation geometry $\leftrightarrow$ algebra

Hopf-algebras

$(A, \mu, \varepsilon, \iota)/R$

Hopf ideals

maps $f^* : B \to A$

Hopf ideal $f^*(ker\varepsilon_B)$

direct limit of

p-divisible groups

$G = (G_n)_n/\mathrm{Spec}(R)$

height

Examples: $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_R$, $(\mu_{p^n,R})_n$, p-torsion in abelian varieties

$R = K$ field of char zero

Cartier's theorem: All Hopf algebras in char 0 are reduced.

Consequence: all ffgs$/K$ are étale.

1st Classification result

$$\{\text{ffgs}/K\} \longrightarrow \left\{\begin{matrix}\text{finite abelian groups } + \\ \text{cts action of } \mathrm{Gal}(\overline{K}/K)\end{matrix}\right\}$$
$$G \longmapsto G(\overline{K})$$

2nd Classification result

$$\left\{\begin{matrix}p\text{-div. groups}/K \\ p\text{-power order}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{finite free } \mathbb{Z}_p\text{-modules} \\ + \text{ cts action of } \mathrm{Gal}(\overline{K}/K)\end{matrix}\right\}$$
$$G \longmapsto T_p(G)$$

R complete local noetherian

G ffgs or p-divisible group, then have the **connected étale sequence**

$1 \to G^0 \to G \to G^{ét} \to 1$

Connected p-divisible groups are the same as formal Lie groups.

dimension of a p-divisible group

$R = k$ perfect field of char p

Frobenius F and Verschiebung V, $FV = p = VF$

The connected-étale sequence splits, $G \cong G^0 \times G^{ét}$.

Further decomposition $G \cong G_{cc} \times G_{cé} \times G_{éc} \times G_{éé}$,

where e.g. $G_{cé}$ means G connected and G^* étale.

$|G_{éé}|$ is prime to p, all others have p-power order.

Classification by Dieudonné theory

$\mathbb{D}(k) := W(k)[F, V]$, the Dieudonné ring, $FV = p = VF$,

$F \cdot x = \sigma(x)F, V\sigma(x) = xV$.

1st Classification result

$$\left\{\begin{matrix}\text{ffgs/K of} \\ p\text{-power order}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite length} \\ \mathbb{D}(k)\text{-modules}\end{matrix}\right\}$$
$$G \longmapsto \mathbb{M}(G)$$

Defining $\mathbb{M}(G)$ is tricky. When $G = G_{cc}$, then $\mathbb{M}(G) = \varinjlim \mathrm{Hom}(G, W_n^m)$, where $W_n^m = \ker(F^m)$ on W_n, the length n Witt vector scheme.

2nd Classification result

$$\{p\text{-div. groups}/K\} \longleftrightarrow \left\{\begin{matrix}\text{finite free } W(k)\text{-modules } + \\ \sigma\text{-linear } F \text{ and } \sigma\text{-antilinear } V\end{matrix}\right\}$$
$$G = (G_n) \longmapsto \mathbb{M}(G) := \varprojlim \mathbb{M}(G_n)$$

R a DVR of mixed characteristic

$K := \mathrm{Frac}(R), C := \widehat{\overline{K}}, S := \mathcal{O}_C$

S-points of a p-div group G/R,

the logarithm.

For ffgs the special fibre functor is not faithful in general. The generic fibre functor is faithful.

For p-div groups, $G \mapsto G_s$ is faithful.

Tate's theorem: For p-div groups, the generic fibre functor is full and faithful.

generic fibre functor

special fibre functor

important concepts

$\longrightarrow$ = specific base