Summary of Talk 1 - Seminar on Abelian Varieties

In talk 1 and 2 of this seminar, we will cover the basic concepts in the theory of abelian varieties. In talk 1, following are topics which will be covered:-

Group schemes

For a scheme S we will recall the notion of an $S-$*group scheme*. It is defined as an $S-$scheme which has a *multiplication, inverse* and an *identity* morphism which satisfies the assoiciative law. All these information is codified in a commutative diagram or equivalently (by *Yoneda lemma*):-

Proposition. An S scheme G is an $S-$group scheme iff the functor $Hom_S(-, G) : Sch_S^{opp} \to Sets$ factors as $Hom_S(-, G) : Sch_S^{opp} \dashrightarrow Groups \hookrightarrow Sets$.

This helps us construct/realize certain objects as group schemes such as $\mathbb{G}_a, \mathbb{G}_m, GL_n$, elliptic curves over a field etc.

Abelian variety

Let k be a field. We define an ***abelian variety***$/k$ to be a smooth(hence a variety), connected, proper group scheme over k. The commutativity of the group law on an abelian variety A follows by considering the morphism $A \times A \to A := (x, y) \mapsto xyx^{-1}y^{-1}$ and applying the rigidity lemma to it:-

Rigidity lemma. Let X, Y, Z be varieties over k with X proper and Z reduced. Let $f : X \times Y \to Z$ be a morphism of varieties s.t. there is a point $y_0 \in Y$ s.t. $f(X \times \{y_0\}) = \{z_0\}$ (i.e. single point). Then there exists a morphism $g : Y \to Z$ s.t. $f = g \circ p_2$ where p_2 is the second projection.

Elliptic curve

Here we will discuss the well known fact that an elliptic curve(i.e. a smooth, connected projective curve of genus 1 with atleast one rational point) is an abelian variety. We will prove this fact from:-

Theorem. For an elliptic curve C/k, the functor $\mathscr{P}ic_{C/k} : Sch_k^{opp} \to Groups$ defined by $T \mapsto \{$family of degree 0 line bundles on $C \times T\}/Pic(T)$ is representable by C itself.

In particular C is an abelian variety and $C(k)$ inherits a group structure where three points $a, b, c \in C(k)$ satisfy $a+b+c = 0$ iff they are collinear. The above theorem is equivalent to the statement that C is its own ***dual abelian variety*** (which will be discussed in talk 4).

Abelian variety over $\mathbb{C}$

Assuming that an abelian variety is projective(this will be proved in talk 2), the *analytification* of an abelian variety over $\mathbb{C}$ is a projective(compact) complex Lie group. From the theory of Lie groups, they are (projective)complex tori. By Serre's GAGA theorem, we know that a projective complex torus admits an algebraic structure s.t. all the associated maps(i.e. the multiplication, inverse etc.) are morphisms of varieties, in particular becomes an abelian variety. It is natural to ask, which complex torus are projective. The answer is the existence of

Riemann form. On a complex torus V/Γ (V a $\mathbb{C}$-vector space and Γ a lattice), a *Riemann form* is an alternating bilinear form $\omega : V \times V \to \mathbb{R}$ s.t.

(a) $\omega(u, v) = \omega(iu, iv)\ \forall u, v \in V$;
(b) $\omega(-, i(-))$ is positive definite;
(c) $\omega(u, v) \in \mathbb{Z}\ \forall u, v \in \Gamma$.

The main result is:-

Theorem. A complex torus is an abelian variety iff it admits a Riemann form.