Homological algebra seminar

Sommersemester 2025

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February 6, 2025

Vorbesprechung. Wednesday, Feb 12, 2025, 9.30 A.M., SR3.

Aim. This Seminar is aimed at Bachelor students who have attended the course Algebra 1. It is intended as a complement to the course Algebra 2 held by Prof. Denis Vogel. Prior knowledge of category theory and module theory will **not** be assumed.

Seminar program. In the context of the category of modules over a commutative ring R, it's possible to encounter several examples of functors, e.g.:

(i) given a multiplicative system $S \subseteq A$, the functor sending an A-module M to its localization $S^{-1}M$;

(ii) given an A-module N, the functor sending an A module M to $N \otimes_A M$;

(iii) given a group G, the functor sending a G-module M to the abelian group M^G of the elements fixed by that action.

Many such functors commute with direct sums and preserve either injective maps (*left exact functors*, as in (iii)), surjective maps (*right exact functors*, as in (ii)), or both (*exact functors*, as in (i)). Homological algebra provides a framework for measuring the deviation of left/right exact functors from exactness in a quantitative way, through *derived functors*. This concept finds its application in various fields of mathematics.

- **Algebraic geometry.** The category of sheaves of abelian groups on a topological space (or an algebraic variety) X behaves in a similar way as the category of A-modules: it's an *abelian category*. The functor sending a sheaf to its global sections is left exact, and yields a derived functor which can be used to study—among other things—intersections of algebraic varieties.
- **Algebraic topology.** Singular/simplicial homology can be interpreted (for nice enough spaces) as the derived functor of the functor of global sections applied to the constant sheaf. This kind of homology is a foundational concept in algebraic topology.
- **Galois theory.** Given a field extension E/L with Galois group G, it's useful to study its Galois cohomology groups $H^*(G, L)$—obtained as the derived functor of (iii); for example, if $E = \bar{L}$, $H^2(G, L)$ counts the number of skew fields whose center is exactly L.

  If $L = \mathbb{R}$, it's possible to prove that $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$, meaning that there are only two skew fields with center $\mathbb{R}$: $\mathbb{R}$ itself and the quaternions $\mathbb{H}$.

Structure. The seminar will be held in **English**. The first two talks of the seminar will consist in an introduction to the theory of modules by the supervisor Giacomo Hermes Ferraro.

For the remaining 10 subjects, each student will choose one and prepare a presentation. For each subject there is a description of the main topics that need to be explored during the presentation and the relevant bibliography. Each presentation should be **between 80 and 90 minutes long**.

Timeline. At least **two weeks before** your scheduled talk, you should discuss with the supervisor, via email or in person, the structure of the talk. At least **one week before** your scheduled talk, you will provide a half-page summary of its content to be reviewed by the supervisor.

Contact. The supervisor is available for any question via email at:
`giacomo.ferraro`[at]`iwr.uni-heidelberg.de`. In person meetings can also be arranged.

Seminar Homepage. Here.

0 A crash course on modules (two lessons)

- A-Modules: definitions and examples.
- Quotients, kernels and direct sums. Homomorphism theorems.
- Finitely generated and finitely presented modules. Nakayama's lemma.
- Bilinear maps and tensor products.
- Base change.
- Localization.

1 Intro to categories ([1, pp. I.1–5, IV.1–2])

- Definition and examples of categories and functors.
- Definition and examples of natural transformations.
- Adjoint functors. Example (adjointedness of $\_ \otimes_A M$ and $\mathrm{Hom}_A(M, \_)$ in $A - \mathbf{Mod}$).

2 Additive categories ([1, V.1–2, Ch. VIII.1–2])

- Limits and colimits. Terminal object and initial object.
- Definition of kernel and cokernel in categories with a null object (e.g. the category of groups).
- Additive categories.
- Epi-mono factorization.

3 Abelian categories ([1, pp. VIII.3–4])

- Definition of abelian categories.
- Definition of a short exact sequence.
- Definition of additive functors and (left/right) exact functors. Examples (tensor functor, hom functor).
- Chain complexes and the five lemma (proof can be done in the category of A-modules).

4 Projective and injective objects ([4, pp. 2.2, 2.3], except Horseshoe lemma)

- Definition of projective and injective objects.
- Equivalency of the definitions of projective modules.
- Definition of projective resolutions and comparison theorem.
- Baer's criterion. Existence of enough injective modules.

5 Derived functors ([4, pp. 1.1, 2.1, 2.4, 2.5])

- Definition of (co)homology, quasi isomorphisms, chain homotopy.
- Definition of δ-functors.
- Horseshoe lemma ([4, Lemma 2.2.8]).
- Left/right derived functors.

6 Tor functor ([4, pp. 3.1, 3.2])

- Definition of flat modules.
- Definition of Tor_*^A. State that $\mathrm{Tor}_*^A(M, N) = \mathrm{Tor}_*^A(N, M)$ (no proof).
- Relation of Tor_1 and torsion.
- Equivalent definition of flatness using Tor ([4, Ex. 3.2.1]).
- Flat base change and localization for Tor.

7 Ext functor ([4, pp. 3.3, 3.4])

- Definition of Ext_A^*. State that $R^*\mathrm{Hom}_A(-, N)(M) = R^*\mathrm{Hom}_A(M, -)(N)$ (no proof).
- Equivalent definitions of projectiveness/injectiveness using Ext ([4, Ex. 2.5.1, 2.5.2]).
- Localization for Ext.
- The extensions functor and its isomorphism with Ext_A^1.

8 Flat modules, projective modules, injective modules ([4, pp. 3.1, 3.3]; see [2, pp. 1.3, 1.4])

- Locality of flatness/projectiveness/injectiveness.
- Computation of Tor and Ext for finite abelian groups.
- Free⇒projective⇒flat and counterexamples.
- In local rings, finitely generated and projective implies free. Use Tor to show the same for finitely presented flat modules.
- An A-module is finitely generated and projective if and only if it is finitely presented and flat.

9 Homological dimension and regular rings ([4, pp. 4.1, 4.4, 4.5])

- Projective dimension of a module. Equivalent definitions using Ext and Tor.
- Global dimension of a Noetherian local ring.
- Regular rings and their properties.
- The Koszul complex.

10 Intro to group cohomology ([3, pp. 1.1–1.6, 1.10])

- Group cohomology via cochains.
- Group cohomology via projective resolutions.
- Tate cohomology.
- Application to cyclic groups.

References

[1] Saunders Mac Lane. *Categories for the working mathematician.* New York: Springer, 1978. URL: `https://math.mit.edu/~hrm/palestine/maclane-categories.pdf`.

[2] Hideyuki Matsumura. "Commutative Algebra". URL: `https://aareyanmanzoor.github.io/assets/matsumura-CA.pdf`.

[3] Romyar Sharifi. "Group and Galois cohomology". URL: `https://www.math.ucla.edu/~sharifi/groupcoh.pdf`.

[4] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. URL: `https://people.math.rochester.edu/faculty/doug/otherpapers/weibel-hom.pdf`.