Summary of Talk 2 (Aim:- Projectivity of abelian varieties) - Seminar on Abelian Varieties

Picard Scheme

We will begin the talk by discussing the existence of Picard scheme. The working hypothesis on a variety will be:-
(*) X/k is a geometrically connected, geometrically reduced and proper variety over the field k and $X(k) \neq \emptyset$.
X/k will be called a variety of type $*$. For a rational point $e \in X(k)$, a ***rigidified line bundle*** is a pair $(\mathscr{L}, i)$ for $\mathscr{L}$ a line bundle on X_S (S a $k-$scheme) s.t. $i : e_S^*\mathscr{L} \simeq \mathcal{O}_S$(if it exists).
One can define a *Picard functor* $\mathscr{P}ic_{X/k,e}$ which takes values in rigidified line bundles modulo isomorphisms. There is a natural isomorphism of functors $\mathscr{P}ic_{X/k,e} \simeq \mathscr{P}ic_{X/k}$. Grothendieck et al proved that
Theorem. $\mathscr{P}ic_{X/k,e}$ is representable by a locally finite type $k-$scheme $Pic_{X/k}$.
We will assume this theorem, without proof, throughout the talk.

Results on line bundles

The following three theorems are fundamental to the study of line bundles on abelian varieties.
Seesaw principal. Suppose X is a variety of type $*$ and Y be a $k-$scheme. Suppose L, M are line bundles on $X \times Y$ s.t. $L_{|X\times\{y\}} \simeq M_{|X\times\{y\}}$ for all points $y \in Y$. Then $\exists$ a line bundle N on Y s.t. $L \simeq M \otimes p_2^*N$.

Theorem of the Cube. Let X, Y, Z be a varieties of type type $*$ and let x_0, y_0, z_0 respectively be rational points on X, Y, Z respectively. Suppose L is a line bundle on $X \times Y \times Z$ s.t. the fibers $L_{|\{x_0\}\times Y\times Z}, L_{|X\times\{y_0\}\times Z}, L_{|X\times Y\times\{z_0\}}$ are trivial. Then L is trivial.

Following is a consequence of the theorem of cube
Theorem of the Square. Let A/k be an abelian variety and $x, y \in A(k)$ be rational points. Let $t_x : A \to A$ be the translation mprphism w.r.t. the point $x \in A(k)$. Then for any line bundle L on A, we have an isomorphism $t_{x+y}^*L \otimes L \simeq t_x^*L \otimes t_y^*L$.

Standard families of line bundles

The theorem of the square motivates one to construct the following homomorphism of group schemes (A being an abelian variety)

$$\phi_L : A \to \hat{A} := Pic^0_{A/k}$$

defined functorially by $x \mapsto t_x^*L \otimes L^{-1}$. Here $\hat{A}$ is called the *dual abelian variety* and it is the identity component of $Pic_{A/k}$.
Under the representability of the Picard functor, the morphism ϕ_L corresponds to the ***Mumford line bundle***

$$\Lambda(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$$

where $m : A \times A \to A$ is the multiplication morphism and p_i is $i-$th projection.
The *universal family of line bundles* on $A \times \hat{A}$ is called the ***Poincaré line bundle***, denoted $\mathscr{P}_A$, which is the restriction of the universal line bundle on $A \times Pic_{A/k}$ to $A \times \hat{A}$. We have the obvious relation

$$(1 \times \phi_L)^*\mathscr{P}_A \simeq \Lambda(L)$$

Projectivity of abelian varieties

The projectivity of abelian varieties will be proved as a consequence of the following four equivalent conditions (for this part one may assume that the base field is algebraically closed):-
Theorem. Let D be a Weil divisor on an abelian variety A/k and $L := \mathcal{O}(D)$ be the associated line bundle. The following are equivalent:-
(a) The set $H := \{x \in A(k) | t_x^*D = D\}$ is a finite;
(b) The set $K(L) := \{x \in A(k) | t_x^*L \simeq L\}$ is finite;
(c) The complete linear system of $L^{\otimes 2}$ is *base-point free* and the corresponding morphism $A \to \mathbb{P}(\Gamma(L^{\otimes 2}))$ is finite;
(d) L is an ample line bundle.

We will apply the theorem as follows:- Choose any affine open set U in A and consider $D := A - U$. In this situation D is pure of codimension 1. We will show that D satisfies condition (a), or equivalently A is a projective variety over k.