# Abelian Varieties

Masters Seminar, Wintersemester 21/22
Dr. Barinder S. Banwait, Prof. Dr. Gebhard Böckle

Tuesdays, 2-4 pm (ct), SR 8
Start Date: Nov. 2, 2021

Vorbesprechung: Di 12.10.21, 15-16 Uhr (st), Raum 0200 (EG)

email: barinder.s.banwait@gmail.com, gebhard.boeckle@iwr.uni-heidelberg.de

## 1. Introduction

An abelian variety over a field $K$ is a smooth, connected, proper $K$-group scheme. Essentially, it is a variety which admits the structure of an algebraic group, which one can show must be commutative. Abelian varieties are natural generalisations of elliptic curves, which are precisely the 1-dimensional abelian varieties, and possess a rich theory comprising algebra, geometry, and arithmetic. While much is known about them, there remain several high-profile open conjectures and problems, and as such this is a subject area of very active current research.

This course will focus on the basic theory of abelian varieties over general fields (not necessarily algebraically closed) as outlined in the first 11 talks of the seminar. The last talk will be an overview of some of the many open problems in the area.

It is hoped that active participation in this course will enable the student to independently study the research literature on abelian varieties.

## 2. Practicalities

1. **Prerequisites.** Students are expected to have a solid background in Algebraic Geometry (including schemes and cohomology) and in Commutative Algebra. Some acquaintance with Elliptic Curves is also assumed.

2. **Requirements of the participants.**

   (1) **Delivery of seminar talk**. The student is expected to deliver a 90-minute talk on their chosen topic. Definitions and results must be stated clearly, and where possible illustrated with concrete examples. Ideally the student will have worked through and understood all proofs; some of these proofs should be presented, though for reasons of time others may have to be omitted.

   (2) **Preparation of handout**. The student is expected to prepare a handout for the other participants containing a summary of their talk, and highlighting the most salient aspects of it.

   (3) **Timelines for preparation of talk and handout**.

(a) Students are expected to arrange a meeting with Dr. Banwait **at least two weeks before** their talk to discuss the proposed talk and to identify any points of confusion. Students are welcome to arrange this meeting more than two weeks before their talk.

(b) In addition to the above, students are expected to arrange a meeting with Dr. Banwait **the week before** their talk, in which the student will summarise the talk they will be giving the following week, as well as discuss and resolve any outstanding points of confusion about the material.

(c) The handout is to be prepared **the week before** the talk, and submitted to Dr. Banwait.

(4) **Mode of delivery**. Students are encouraged to deliver their talk in person to the other participants of the seminar, either on the chalkboard or with a Beamer or slideshow presentation. However, in case the student is not physically present in Heidelberg when giving their talk, they may deliver their talk online via Webex.

(5) **Language of talk and handout**. Students may deliver their talk in either English or German. The language of the handout should be the same as the language the talk is given in.

3. **Recommended reading.** The canonical reference for this subject remains Mumford's *Abelian varieties* [Mum74]. The notes of courses given by Bhargav Bhatt [Bha17] and Brian Conrad [Con15] are also excellent references. Milne's notes [Mil08] as well as the book-in-progress by Edixhoven, van der Geer, and Moonen [EGM21] are valuable references, though the reader should bear in mind that they are incomplete in places.

A survey of the notable open conjectures in the arithmetic of abelian varieties may be found in Cantoral-Farfán's expository article [CF16]. See also [CFC19].

4. **Student Evaluation.** Students will be evaluated based on the clarity of both the talk as well as the handout.

## 3. Talks

0. **Overview of the course (Seminarvorbesprechung).** Oct. 12

The course content will be sketched in summary. The remaining talks will be distributed amongst the participants.

1. **Definitions and basic examples.** Nov. 2

Group schemes and varieties. Definition of abelian variety. Rigidity, and elliptic curves as 1-dimensional abelian varieties. Riemann forms, complex tori, and their link with complex abelian varieties (GAGA). Commutativity of abelian varieties.

*References*: [Con15], Section 1. [EGM21], Chapter 1.

2. **Line bundles and divisors on abelian varieties.** Nov. 9
Line bundles, Weil and Cartier divisors on abelian varieties. Seesaw theorem, Theorem of the cube and square. The Mumford and Poincaré line bundles. Ample divisors yield polarizations. The kernel subscheme is a subgroup scheme. Non-degenerate line bundles. Abelian varieties are projective.
*References*: [Con15], Section 3. [EGM21], Chapter 2.

3. **Torsion and isogenies.** Nov. 16
Definition of Isogeny, and basic properties. Multiplication by $n$ is an isogeny. Degree of $[n]$, and structure of $A[n](\bar{k})$. Frobenius and Verschiebung, and application to abelian varieties in positive characteristic. The Tate module.
*References*: [Con15], Section 4. [EGM21], Chapter 5.

4. **The dual abelian variety.** Nov. 23
Construction of the dual abelian variety via representability of the (relative) Picard functor. Characterization of $\phi_{\mathcal{L}}$, and the Néron-Severi group. Cohomology of the Poincaré bundle, and applications to dual morphisms and dual isogenies.
*References*: [Con15], Sections 5 and 7. [EGM21], Chapter 6 (excluding Section 2).

5. **Polarizations and Weil Pairings.** Nov. 30
Definitions, examples, and basic examples. Cartier duality. Existence of polarizations, and Zarhin's trick. Every abelian variety over an algebraically closed field is isogenous to one that admits a principal polarization.
*References*: [Con15], Section 8. [EGM21], Chapter 11. [Mil08] Chapter 1 Sections 11 and 13.

6. **The endomorphism ring.** Dec. 7
Definitions and basic results. Poincaré splitting, simple abelian varieties, and isogeny decomposition. The category of abelian varieties up to isogeny. Injectivity of $\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$ and corollaries. The characteristic polynomial of endomorphisms. Independence of $\ell$ of the characteristic polynomial of the induced endomorphism to the $\ell$-adic Tate module, and corollaries. Statement of Tate conjecture for abelian varieties over finitely generated fields of characteristic different from $\ell$.
*References*: [Con15], Section 7.6. [EGM21], Chapter 12 Sections 1 and 2. [Mil08] Chapter 1 Section 10. [Mum74] Chapter 4.

7. **The Albert classification.** Dec. 14
Rosati involution, and its positivity. The Albert Classification. Numerical restrictions on the endomorphism algebra and the Néron-Severi rank. Detailed examples of the different types in dimensions 1, 2 and 3. Brief remarks about abelian varieties with complex multiplication.
*References*: [Con15], Section 7.6. [EGM21], Chapter 12 Sections 3 and 4. [Mil08] Chapter 1 Section 14. [Mum74] Chapter 4.

8. **The Mordell-Weil Theorem.** Dec. 21
Statement and proof of Weak Mordell-Weil theorem. Proof of Mordell-Weil from Weak Mordell-Weil plus heights. Brief overview of Weil's height machine (proofs not required).
*References*: [Con15], Sections 9 and 10.

9. **Jacobian varieties.** Jan. 11

Definition of Jacobian of a curve. Basic results and examples. Comparison with $d^{\text{th}}$ symmetric power of curve. Universal line bundles and the theta divisor. Explicit example of theta divisor for a genus 3 curve.

*References*: [EGM21], Chapter 14, Sections 1–5. [Mil08], Chapter 3, Section 1–5.

10. **The Jacobian as Albanese, and applications.** Jan. 18

The Albanese variety, and relation to Jacobian. Every abelian variety is a factor of a Jacobian. Statement and proof of Torelli's theorem. Statement and discussion of the criterion of Matsusaka-Ran.

*References*: [EGM21], Chapter 14, Sections 6–9. [Mil08], Chapter 3 Section 6, 10–13.

11. **Abelian varieties over finite fields.** Jan. 25

Frobenius eigenvalues. Hasse-Weil bound for curves. Tate's theorem - statement and corollaries (proof not necessary). Honda-Tate theory. Time permitting, Deuring's criterion on trace of Frobenius for elliptic curves ([EGM21, Theorem 16.64], [Wat69, Theorem 4.1]).

*References*: [EGM21], Chapter 16, Sections 1–5. [Wat69], Chapter 4.

12. **Open problems - The Hodge, Tate, Mumford-Tate and algebraic Sato-Tate conjectures.** Feb. 1

Hodge structures and the Mumford-Tate group. Computation of possible Mumford-Tate groups of elliptic curves. Brief discussion of Sato-Tate groups. Statements of the four open conjectures, links between them, known results, and the state of the art for each.

*References*: [CF16], [CFC19].

## References

[Bha17] Bhargav Bhatt. Topics in Algebraic Geometry I - Abelian Varieties. Notes written by Matt Stevenson, available at `http://www-personal.umich.edu/~stevmatt/abelian_varieties.pdf`, 2017.

[CF16] Victoria Cantoral-Farfán. A survey around the Hodge, Tate and Mumford-Tate conjectures for abelian varieties. *arXiV preprint*, 2016. `https://arxiv.org/abs/1602.08354`.

[CFC19] Victoria Cantoral-Farfán and Johan Commelin. The Mumford-Tate conjecture implies the algebraic Sato-Tate conjecture of Banaszak and Kedlaya. `https://arxiv.org/abs/1905.04086`, 2019.

[Con15] Brian Conrad. Abelian Varieties. Notes written by Tony Feng, available at `http://virtualmath1.stanford.edu/~conrad/249CS15Page/handouts/abvarnotes.pdf`, 2015.

[EGM21] Bas Edixhoven, Gerard van der Geer, and Ben Moonen. Abelian Varieties. In progress, available at `http://van-der-geer.nl/~gerard/AV.pdf`, 2021.

[Mil08] James S. Milne. Abelian Varieties (v2.00). Available at `www.jmilne.org/math`, 2008.

[Mum74] David Mumford. *Abelian varieties*, volume 3. Oxford university press, 1974. Available at `https://wstein.org/edu/Fall2003/252/references/mumford-abvar/Mumford-Abelian_Varieties.pdf`.

[Wat69] William C Waterhouse. Abelian varieties over finite fields. In *Annales scientifiques de l'École normale supérieure*, volume 2, pages 521–560, 1969. Available at `http://www.numdam.org/article/ASENS_1969_4_2_4_521_0.pdf`.