ON THE ENDOMORPHISM ALGEBRA OF ABELIAN VARIETIES ASSOCIATED WITH HILBERT MODULAR FORMS

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Abstract. In this article, we will generalize an explicit formula proved by Quer for the Brauer class of the endomorphism algebra of abelian varieties associated to modular forms of weight 2 to the case of Hilbert modular forms of parallel weight 2, under the condition that the degree of the base field over $\mathbb{Q}$ is an odd number.

1. Introduction

Let K be a totally real number field such that $[K:\mathbb{Q}]$ is odd and let f be a non-CM (Hilbert) newform of parallel weight 2, level N where N is an ideal of O_K and finite central character ϵ. It is well-known that in this case one can use Shimura curves to construct an abelian variety A_f over K associated with f. Let $T_\mathfrak{p}$ be the Hecke operator at $\mathfrak{p}$ and $a_\mathfrak{p}$ the eigenvalue of $T_\mathfrak{p}$ acting on f. Then $E = \mathbb{Q}(\{a_\mathfrak{p}\}_\mathfrak{p})$ is a number field called the Hecke field of f. The abelian variety A_f/K is of dimension $d = [E:\mathbb{Q}]$ and hence its ℓ-adic Tate module (after tensoring with $\mathbb{Q}$) V_ℓ is of dimension $2d$ over $\mathbb{Q}_\ell$. One can define an E-structure on this Tate module by letting $a_\mathfrak{p}$ act via the Hecke operator at $\mathfrak{p}$. This turns V_ℓ into a rank 2 free module over $E \otimes_\mathbb{Q} \mathbb{Q}_\ell$ endowed with a continuous G_K-action. This is the Galois representation associated with f which after a choice of basis can be written as:

$$\rho_{f,\ell} : G_K \to \mathrm{Aut}_E(V_\ell) \simeq \mathrm{GL}_2(E \otimes_\mathbb{Q} \mathbb{Q}_\ell)$$

It is easy to see that this Galois representation is unramified outside ℓN and for any unramified prime ideal $\mathfrak{p}$, the Eichler-Shimura relation implies that the characteristic polynomial of $\rho_{f,\ell}(\mathrm{Frob}_\mathfrak{p})$ is equal to $X^2 - a_\mathfrak{p}X + \epsilon(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})$. In particular, $\{\rho_{f,\ell}\}_\ell$ is a compatible family of Galois representations.

In appendix B of [7], Nekovář studies the image of the Galois representation associated with a Hilbert modular form (not necessarily of weight 2) and generalizes results of Ribet [9] and Momose [6] to this case. He constructs a division algebra D over a subfield F of the Hecke field E which describes the image up to p-adic openness. In the special case where one knows there is an abelian variety associated with f (in particular f is of parallel weight 2) F is equal to the center of the algebra $X := \mathrm{End}_{\overline{\mathbb{Q}}}(A_f) \otimes_\mathbb{Z} \mathbb{Q}$ [7, B.4.11] and since A_f is of GL_2-type over K and f is non-CM, it is a Ribet-Pyle abelian variety, i.e. $E \simeq \mathrm{End}_K(A_f) \otimes_\mathbb{Z} \mathbb{Q}$ is a maximal subfield of the simple algebra X [5, Propposition 3.1]. Moreover, D and X have the same class in the Brauer group of F [7, B.4.11] and D is the Mumford-Tate group of A_f . It is natural to ask, if one can find explicit formulas for this class in the Brauer group in terms of the Hecke eigenvalues of f. When $K = \mathbb{Q}$, Quer was able to prove such a formula [8]. This was later generalized to higher weights in [4] for the endomorphism ring of the motive associated with the form.

Quer's result is about classical modular form. In this paper, we will generalize his result to Hilbert modular forms of parallel weight 2 over any odd degree extension K of $\mathbb{Q}$ assuming that the central character ϵ is trivial. In section 2 we will generalize a theorem of Ribet [10, Theorem 5.5] to our situation. This is the main arithmetic input in the proof of Quer's formula. Ribet's proof works without much change but we will repeat the arguments for the convenient of the reader and because this does not seem to be written down in the literature in this case. In section 3 we will generalize [10, Theorem 5.6] using the work of Chi [2]. Here some of the Galois cohomology computations become more complicated due to the fact that our base field K is not contained in the field F, whereas the case of classical modular forms. Therefore one needs to carefully go up and down between some base fields to be able to carry out the computations. Finally, in section 4 we are able to prove Quer's formula in our setting.

Acknowledgments. I would like to thank Gebhard Böckle, Andrea Conti and Judith Ludwig for many valuable discussions about this work. This research was supported by the Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 Geometry and Arithmetic of Uniformized Structures, project number 444845124.

2. Endomorphism Ring and Galois representation

The main goal of this section is to generalize [10, Theorem 5.5.] to the case of Hilbert modular forms. Ribet uses Faltings' theorem on isogenies (Tate conjecture) to relate the endomorphism algebra X to the Tate module. We will do the same thing and closely follow Ribet's arguments. We will keep the assumptions and the notations from the first paragraph of the introduction.

Choose a prime number ℓ that splits completely in E. Then one has d different embeddings $\sigma : E \to \mathbb{Q}_\ell$. Let M be a finite Galois extension of K such that all of the endomorphisms of A_f are defined over M. Now by Faltings' isogeny theorem one has

$$X \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell) \tag{1}$$

Remember that V_ℓ also carries an E-structure through the Hecke action. Every embedding σ of E into $\mathbb{Q}_\ell$ gives a $E \otimes \mathbb{Q}_\ell$ -module structure on $\mathbb{Q}_\ell$ with respect to which we can define

$$V_\sigma = V_\ell \otimes_{E\otimes\mathbb{Q}_\ell,\sigma} \mathbb{Q}_\ell$$

which is a $\mathbb{Q}_\ell$-subspace of V_ℓ of dimension 2 that is invariant under the action of G_K. Now note that $a \in E$ acts on V_σ via multiplication by $\sigma(a) \in E$ hence for two different embeddings σ and τ, V_σ and V_τ have trivial intersection as subspaces of V_ℓ. This (together with obvious dimension reason) gives a decomposition

$$V_\ell = \bigoplus_{\sigma:E\hookrightarrow\mathbb{Q}_\ell} V_\sigma$$

of $\mathbb{Q}_\ell[G_K]$-modules. The following lemma will be useful later.

Lemma 1. *For each embedding σ one has $\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\sigma) = \mathbb{Q}_\ell$. In particular, V_σ is absolutely irreducible as a G_M-representation.*

Proof. From (1) one has $X \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$. Since E is a maximal subfield of X, taking the centralizer of $E \otimes \mathbb{Q}_\ell$ of both sides one gets

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \mathrm{End}_{E\otimes\mathbb{Q}_\ell[G_M]}(V_\ell)$$

which means

$$\oplus_\sigma \mathbb{Q}_\ell = \oplus_\sigma \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\sigma)$$

which implies the first part. Since V_σ is semi-simple by Faltings' proof of the Tate conjecture, irreducibility follows immediately. □

For every prime $\mathfrak{p}$ of O_K not dividing ℓN, recall that $\mathrm{Frob}_\mathfrak{p}$ action on V_ℓ has characteristic polynomial

$$X^2 - a_\mathfrak{p} X + \epsilon(\mathfrak{p}) N(\mathfrak{p}) \in E[X]$$

therefore, for every embedding $\sigma : E \to \mathbb{Q}_\ell$ one has

$$\mathrm{tr}(\mathrm{Frob}_\mathfrak{p} \circlearrowright V_\sigma) = \sigma(a_\mathfrak{p}) \in \mathbb{Q}_\ell$$

Restricting the compatible family of Galois representation to G_M, one gets another compatible family, namely for every finite place v of M not dividing ℓN there is $t_v \in E$ such that

$$\mathrm{tr}(\mathrm{Frob}_v \circlearrowright V_\sigma) = \sigma(t_v) \in \mathbb{Q}_\ell$$

Let $\Sigma_{\ell N}$ be the set of finite places of M not dividing ℓN and $L = \mathbb{Q}(t_v : v \in \Sigma_{\ell N}) \subset E$. Then one has the following:

Lemma 2. *The center of the algebra* $\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$ *is* $L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Proof. First note that by Faltings' theorem

$$E \otimes \mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell) \subset \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$$

and since $E \otimes \mathbb{Q}_\ell$ centralizes itself, it should contain the center of $\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$.

By Semi-simplicity of V_ℓ, one can see that V_σ and V_τ are isomorphic as G_M-representations if and only if they have the same Frob_v traces for all places v of M not dividing ℓN or equivalently σ and τ agree on L. Now let $\gamma : L \to \mathbb{Q}_\ell$ be an embedding and define

$$V_\gamma = \oplus_{\sigma|_L = \gamma} V_\sigma$$

So one has the decomposition $V = \oplus V_\gamma$ and also since there is clearly no non-trivial endomorphism from one V_γ to another one also has the decomposition

$$\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell) = \oplus_\gamma \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\gamma)$$

Now let $a \in L$. Then a acts on V_σ by $\sigma(a)$ hence it acts on the whole subspace V_γ by the scalar $\gamma(a) \in \mathbb{Q}_\ell$ which means (because of the decomposition above) it's in the center of $\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$. So the E-algebra structure on $\mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell)$ induces this L-algebra structure on $Z(\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell))$ which means it's enough to prove

$$Z(\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)) \simeq L \otimes \mathbb{Q}_\ell$$

as L-algebras. This is easy to check:

$$Z(\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)) = Z(\oplus_\gamma \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\gamma)) = \oplus_\gamma Z(\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\gamma)) \simeq \oplus_\gamma \mathbb{Q}_\ell = L \otimes \mathbb{Q}_\ell$$

□

Corollary 1. *L is the center of X, i.e. $L = F$.*

Proof. Recall that from Faltings' isogeny theorem we had

$$X \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)$$

Now from the last lemma:

$$L \otimes \mathbb{Q}_\ell = Z(\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\ell)) = Z(X \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = F \otimes \mathbb{Q}_\ell$$

which implies $F = L$. □

Lemma 3. *If $\sigma, \tau : E \to \mathbb{Q}_\ell$ are embeddings that agree on F then there exists a character $\phi : G_K \to \mathbb{Q}_\ell^\times$ such that $V_\sigma \simeq V_\tau \otimes \phi$ as representation of G_K.*

Proof. From the proof of lemma 2 we know that since σ and τ agree on $F = L$, V_σ and V_τ are isomorphic as representations of G_M. So we can choose two bases for V_σ and V_τ such that the homomorphisms $\rho_\sigma : G_K \to GL_2(\mathbb{Q}_\ell)$ and $\rho_\tau : G_K \to GL_2(\mathbb{Q}_\ell)$ associated with V_σ and V_τ are equal on G_M. Now define

$$\phi(g) := \rho_\sigma^{-1}(g)\rho_\tau(g)$$

A priori ϕ is just a map $\phi : G_K \to GL_2(\mathbb{Q}_\ell)$ which is trivial on G_M. We want to prove that it is actually a homomorphism with values in the center (hence actually a character).

Let $g \in G_K$ and $h \in G_M$. Note that $\rho_\sigma(h) = \rho_\tau(h)$ and $\rho_\sigma(ghg^{-1}) = \rho_\tau(ghg^{-1})$ since G_M is normal in G_K. Now the following computation shows that $\phi(g) = \rho_\sigma^{-1}(g)\rho_\tau(g)$ commutes with $\rho_\tau(h)$:

$$\rho_\sigma^{-1}(g)\rho_\tau(g)\rho_\tau(h) = \rho_\sigma^{-1}(g)\rho_\tau(gh) = \rho_\sigma^{-1}(g)\rho_\tau(ghg^{-1})\rho_\tau(g)$$
$$= \rho_\sigma(g^{-1})\rho_\sigma(ghg^{-1})\rho_\tau(g) = \rho_\sigma(h)\rho_\sigma^{-1}(g)\rho_\tau(g) = \rho_\tau(h)\rho_\sigma^{-1}(g)\rho_\tau(g)$$

Now since $\mathrm{End}_{\mathbb{Q}_\ell[G_M]}(V_\tau) = \mathbb{Q}_\ell$ we are done. □

Corollary 2. *Using the notation of the last lemma, $\phi^2 = \frac{\sigma_\epsilon}{\tau_\epsilon}$ and for any prime $\mathfrak{p}$ of K of good reduction for A_f, one has*

$$\sigma(a_\mathfrak{p}) = \phi(\mathrm{Frob}_\mathfrak{p})\tau(a_\mathfrak{p})$$

Proof. Note that $V_\sigma \simeq V_\tau \otimes \phi$. Taking determinants of both sides one gets the first part and taking trace one gets the second part. □

Ribet also proves that in the $K = \mathbb{Q}$ case, the field F is generated by $\{a_p^2/\epsilon(p)\}_{p \nmid N}$. This is also true in our case. In fact, by [7, B.4.11] F is exactly the field fixed by inner-twists and the above result is known in much more generality in this context by the results of [3].

Proposition 1 ([3], Corollary 4.12)**.** *The field F is generated over $\mathbb{Q}$ by numbers $a_\mathfrak{p}^2/\epsilon(\mathfrak{p})$ for $\mathfrak{p} \nmid N$.*

Now let $\sigma \in G_K$, then σ acts on the $\overline{\mathbb{Q}}$-endomorphisms of A_f by $\sigma(\phi)(x) := \sigma(\phi(\sigma^{-1}x))$ and this linearly extends to an action on X. E is clearly invariant under the action of G_K on X (we are identifying E with the maximal subfield of X). Since this is an automorphism of F-algebras, By the Skolem-Noether theorem the action of σ is given by conjugation by some element $\alpha(\sigma) \in X$. Since E is invariant under the Galois action, $\alpha(\sigma)$ commutes with E and therefore $\alpha(\sigma) \in E$ because E is a maximal subfield and hence its own centralizer. The next theorem relates the map α which is of geometric (motivic) nature to the (automorphic) data of Hecke eigenvalues.

Theorem 1. *For every $\sigma \in G_K$ one has $\alpha(\sigma)^2/\epsilon(\sigma) \in F^\times$. Moreover, for every prime ideal $\mathfrak{p}$ of O_K away from ℓN, if $a_\mathfrak{p} \neq 0$ then $\alpha(\mathrm{Frob}_\mathfrak{p}) \equiv a_\mathfrak{p}$ modulo $F^\times$.*

Proof. As usual, let ℓ be a prime number that splits completely in E. It enough to prove that for every pair of embeddings σ and τ of E in $\mathbb{Q}_\ell$ that agree on F one has $\sigma(\alpha^2/\epsilon) = \tau(\alpha^2/\epsilon)$.

Now if σ and τ agree on F then by lemma 3 there exist a character $\phi : G_K \to \mathbb{Q}_\ell^\times$ such that $V_\sigma \simeq V_\tau \otimes \phi$ as representation of G_K. In particular, it implies that as 1-dimensional representation of G_K one has

$$\mathrm{Hom}_{\mathbb{Q}_\ell[G_M]}(V_\sigma, V_\tau) \simeq \phi$$

Also note that if σ and τ don't agree on F then they are not isomorphic as G_M-representation hence

$$\mathrm{Hom}_{\mathbb{Q}_\ell[G_M]}(V_\sigma, V_\tau) = 0$$

Therefore we can completely understand $X \otimes \mathbb{Q}_\ell$:

$$\mathrm{End}^0_M(A_f) \otimes \mathbb{Q}_\ell \simeq \mathrm{End}_{\mathbb{Q}_\ell[G_M]}(\oplus V_\gamma) = \oplus_{\sigma,\tau} \mathrm{Hom}_{\mathbb{Q}_\ell[G_M]}(V_\sigma, V_\tau)$$

Now remember that on the LHS, $g \in G_K$ acts via conjugation by $\alpha(g)$. Hence, it acts on V_σ and V_τ by $\sigma(\alpha(g))$ and $\tau(\alpha(g))$ respectively. Now assume that σ and τ agree on F. Then g acts on $\mathrm{Hom}_{\mathbb{Q}_\ell[G_M]}(V_\sigma, V_\tau)$ by $\sigma(\alpha(g))/\tau(\alpha(g))$. On the other hand as a representation of G_K this is just ϕ so $\sigma(\alpha(g))/\tau(\alpha(g)) = \phi(g)$. Since $\phi^2 = \frac{\sigma_\epsilon}{\tau_\epsilon}$ one deduces that $\sigma(\alpha^2/\epsilon) = \tau(\alpha^2/\epsilon)$ and the result follows.

For the second part, first notice that

$$\phi(\mathrm{Frob}_\mathfrak{p}) = \sigma(\alpha(\mathrm{Frob}_\mathfrak{p}))/\tau(\alpha(\mathrm{Frob}_\mathfrak{p})) = \sigma(a_\mathfrak{p})/\tau(a_\mathfrak{p})$$

therefore

$$\sigma(\alpha(\mathrm{Frob}_\mathfrak{p})/a_\mathfrak{p}) = \tau(\alpha(\mathrm{Frob}_\mathfrak{p})/a_\mathfrak{p})$$

which implies the result. □

3. Twisted Algebras and Galois Cohomology

The goal of this section is to prove an analogue of [10, Theorem 5.6] in our setting. Ribet uses a result of Chi to prove this theorem. In [2], Chi studies the twists of a central simple algebra by a 1-cocycle. We need to review some of his results and generalize some of those to our setting.

First note that the endomorphism ring $\mathrm{End}_{\overline{\mathbb{Q}}}(A_f)$ acts on the space of differential 1-forms on $A_f/\overline{\mathbb{Q}}$ (which we denote by $\Omega^1_{\overline{\mathbb{Q}}}$) via pull back. For an endomorphism ϕ and a 1-form ω we use the usual notation $\phi^*\omega$ for this action. This action linearly extends to an action of X on this space and we use the same notation for this action as well. Also, note that for any $\sigma \in G_K$ and $\phi \in X$ one has

$$\sigma(\phi^*\omega) = (\sigma\phi)^*(\sigma\omega) = (\alpha(\sigma) \cdot \phi \cdot \alpha(\sigma)^{-1})^*(\sigma\omega) \tag{2}$$

For σ and τ in G_K, define $c_\alpha(\sigma, \tau) := \alpha(\sigma)\alpha(\tau)\alpha(\sigma\tau)^{-1}$. This commutes with every element in X so it lands in F. Therefore α gives a well-defined group homomorphism

$$\alpha_K : G_K \to \frac{E^\times}{F^\times}$$

Let α_{FK} be the restriction of α_K to G_{FK}. We sometimes use the same notation to denote the composition of this map with the canonical map to $(EK)^\times/(FK)^\times$:

$$\alpha_{FK} : G_{FK} \to \frac{E^\times}{F^\times} \to \frac{(EK)^\times}{(FK)^\times}$$

Let $X_{FK} := X \otimes_F FK$. This is an algebra over FK. Note that every element in FK is a sum of the form $\sum_i f_i k_i$ for $f_i \in F$ and $k_i \in K$ so X_{FK} is generated by pure tensors of the form $\sum_i \phi_i \otimes k_i$ for $k_i \in K$.

As in [2] one can look at the twist of this algebra with (the 1-cocycle defined by) α which we denote by $X_{FK}(\alpha_{FK})$ following Chi.

Proposition 2 ([2], Proposition 1.1)**.** *One has*

$$\dim_{FK} X_{FK}(\alpha_{FK}) = \dim_{FK} X_{FK} = dim_F X$$

and Moreover

$$X_{FK}(\alpha_{FK}) \otimes_{FK} \overline{\mathbb{Q}} \simeq X_{FK} \otimes_{FK} \overline{\mathbb{Q}}$$

Therefore, $X_{FK}(\alpha_{FK})$ is a central simple FK-algebra.

One can also view X_{FK} as a K-algebra and twist it with α_K instead to get the K-algebra $X_{FK}(\alpha_K)$. Let us recall the definition of this algebra. First for any $\sigma \in G_K$ we define the twisted action of σ on $X_{FK} \otimes_K \overline{\mathbb{Q}}$ as follow. On pure tensors of the form $\phi \otimes k \otimes \lambda$ for $\phi \in X$, $k \in K$ and $\lambda \in \overline{\mathbb{Q}}$ we define:

$$tw(\sigma) \cdot (\phi \otimes k \otimes \lambda) := \alpha(\sigma)\phi\alpha(\sigma)^{-1} \otimes k \otimes \sigma(\lambda)$$

Note that $k = \sigma(k)$ in the above expression. Now we define:

$$X_{FK}(\alpha_K) := (X_{FK} \otimes_K \overline{\mathbb{Q}})^{tw(G_K)}$$

This K-algebra also has the structure of an FK-algebra via $a \cdot \sum \psi_i \otimes \lambda_i := \sum a\psi_i \otimes \lambda_i$ for $a \in FK$, $\psi_i \in X_{FK}$ and $\lambda_i \in \overline{\mathbb{Q}}$.

Proposition 3 ([2], Proposition 1.2)**.** *One has $X_{FK}(\alpha_{FK}) \simeq X_{FK}(\alpha_K)$ as FK-algebras.*

This implies that $X_{FK}(\alpha_K) = (X_{FK} \otimes_K \overline{\mathbb{Q}})^{tw(G_K)}$ is also a central simple FK-algebra. From now on we simply write $X_{FK}(\alpha)$ for this central simple algebra.

E is a subfield of X_{FK}. Let L be a maximal subfield of X_{FK} containing E. Then L contains EK as well. So one can look at α_{FK} as a group homomorphism

$$\alpha_{FK} : G_{FK} \to \frac{L^\times}{(FK)^\times}$$

which has values in E. Now one can apply [2, Proposition 2.4] to get:

$$X_{FK}(\alpha_{FK}) \otimes_{FK} \mathrm{End}_{FK} L \simeq X_{FK} \otimes_{FK} \mathrm{End}_{FK} L(\alpha_{FK})$$

So in the $Br(FK)$ one has

$$[X_{FK}(\alpha)] = [X_{FK}] + [\mathrm{End}_{FK} L(\alpha_{FK})]$$

From this point onward, we assume that the central character ϵ of f is trivial. In the general case, one also needs to carry the 2-cocycle $c_\epsilon = [\mathrm{End}_{FK} L(\epsilon)]$ in the calculations as in [8] which complicates some of the computations. Having this assumption, now we can prove:

Lemma 4. *Assuming ϵ is trivial, the order of $[X_{FK}(\alpha)]$ in $Br(FK)$ divides 2.*

Proof. So far we proved

$$[X_{FK}(\alpha)] = [X_{FK}] + [\mathrm{End}_{FK} L(\alpha_{FK})]$$

in $\mathrm{Br}(FK)$. By [7, Proposition B.4.12] we know that X and hence X_{FK} have Schur index dividing 2. Also from theorem 1 we know that $\alpha^2 = \epsilon$ modulo $F^\times$. Applying [2, Proposition 2.2] we get:

$$2 \cdot [\mathrm{End}_{FK} L(\alpha_{FK})] = [\mathrm{End}_{FK} L(\alpha_{FK}^2)] = [\mathrm{End}_{FK} L(\epsilon)]$$

Since $\epsilon = 1$ we are done. □

From section 2 of [2] we know that $[\mathrm{End}_{FK} L(\alpha_{FK})]$ in $Br(FK) = H^2(G_{FK}, \overline{\mathbb{Q}})$ is the same as the image of the cohomology class defined by α in $H^1(G_{FK}, PGL_n(\overline{\mathbb{Q}}))$ under the connecting homomorphism

$$\delta : H^1(G_{FK}, PGL_n(\overline{\mathbb{Q}})) \to H^2(G_{FK}, \overline{\mathbb{Q}})$$

where $n = [L : FK]$. More concretely, one can view every $\ell \in L$ as an FK-linear endomorphism $\ell : L \to L$ given by multiplication by ℓ. So every ℓ can be viewed as an $n \times n$ matrix with FK-entries. Now viewing every $\alpha(\sigma) \in E$ as such a matrix, conjugation by this matrix gives an element in $PGL_n(FK) \subset PGL_n(\overline{\mathbb{Q}})$. This gives 1-cocycle with $PGL_n(FK)$ or rather with $PGL_n(\overline{\mathbb{Q}})$ values that is invariant under the G_{FK} action. Since the connecting homomorphism δ sends a 1-cocycle f to $f(\sigma)\sigma(f(\tau))f(\sigma,\tau)^{-1}$ one concludes:

Corollary 3. *Let $c_\alpha(\sigma,\tau) = \alpha(\sigma)\alpha(\tau)\alpha(\sigma\tau)^{-1}$ be a 2-cocycle for the trivial action of G_K on $F^\times$. Then the image of $[c_\alpha]$ under the sequence*

$$H^2(G_K, F^\times) \xrightarrow{\mathrm{res}} H^2(G_{FK}, F^\times) \xrightarrow{\iota_*} H^2(G_{FK}, \overline{\mathbb{Q}}^\times)$$

is exactely the class of $[X_{FK}(\alpha)]$ in $H^2(G_{FK}, \overline{\mathbb{Q}}) = Br(FK)$

Corollary 4. *In $Br(FK)$ one has:*

$$[X_{FK}(\alpha)] = [X_{FK}] + \iota_*(\mathrm{res}([c_\alpha]))$$

Our next goal is to prove that $X_{FK}(\alpha)$ is trivial in the Brauer group. The main ingredient is the next proposition.

Proposition 4. *$X_{FK}(\alpha)$ acts (linearly) on Ω^1_K.*

Proof. First, we defined an action of $X_{FK} \otimes \overline{\mathbb{Q}}$ on $\Omega^1_{\overline{\mathbb{Q}}}$ by extending the action of X linearly, namely we define:

$$(\phi \otimes k \otimes \lambda)^*\omega := k\lambda\phi^*\omega$$

for $\phi \in X$, $k \in K$ and $\lambda \in \overline{\mathbb{Q}}$. Now using (2) one easily sees that for any $\sigma \in G_K$ and $\psi \in X_{FK} \otimes \overline{\mathbb{Q}}$:

$$\sigma(\psi^*\omega) = (tw(\sigma)\cdot\psi)^*\sigma\omega$$

which means that if ψ is invariant under the twisted Galois action and ω is invariant under the usual Galois action, then $\psi^*\omega$ is also invariant. This means that $X_{FK}(\alpha)$ acts on Ω^1_K. □

Proposition 5. *$X_{FK}(\alpha) \in Br(FK)$ is trivial.*

Proof. Let $X_{FK}(\alpha) = M_n(D)$ for some division algebra D over FK of dimension s^2. By corollary 4 one has $s|2$. Now $\dim X_{FK}(\alpha) = n^2s^2$ which should be equal to the dimension of X over F therefore $ns = [E : F]$. By the last proposition Ω^1_K is a $M_n(D)$-module. So there is a D-vector space W such that $\Omega^1_K \simeq W^n$. The dimension of Ω^1_K over K is equal to the dimension of the abelian variety A_f which is $[E : \mathbb{Q}]$. Hence

$$s^2 = \dim_{FK} D|\dim_{FK} W = \frac{[E : \mathbb{Q}]}{n[FK : K]} = \frac{ns[F : \mathbb{Q}]}{n[F : F \cap K]} = s[F \cap K : \mathbb{Q}]$$

This implies $s|[F \cap K : \mathbb{Q}]$ but since $s|2$ and $[K : \mathbb{Q}]$ is odd, one has $s = 1$. □

From proposition 5 and corollary 4 and the fact that $[X_{FK}] \in Br(FK)$ has order dividing 2, one deduces:

Corollary 5. *In $Br(FK)$ one has*

$$[X_{FK}] = \iota_*(\mathrm{res}([c_\alpha]))$$

Now we need to go down from FK to F to compute the class $[X]$ in $Br(F)$ using α. We can use the corestriction map to do so. First note that by the last corollary we know that following the below diagram, the image of $[c_\alpha]$ in $H^2(G_{FK}, \overline{F}^*)$ is $[X_{FK}]$ which is the image of $[X]$ under the restriction.

$$\begin{array}{ccc} [c_\alpha] \in H^2(G_K, F^*) & \xrightarrow{\text{res}} & H^2(G_{FK}, F^*) \\ & & \downarrow \iota_* \\ & \xleftarrow{\text{cor}} & \\ [X] \in H^2(G_F, \overline{F}^*) & \xrightarrow[\text{res}]{} & H^2(G_{FK}, \overline{F}^*) \end{array}$$

This means that

$$\iota_*(\mathrm{res}([c_\alpha])) = \mathrm{res}([X])$$

On the other hand, $\mathrm{cor} \circ \mathrm{res} = [FK : F] = [K : F \cap K]$ which is an odd integer. Since X has order dividing 2 in the Brauer group, $\mathrm{cor}(\mathrm{res}([X])) = X$.

Finally, we can conclude the generalization of [10, Theorem 5.6] to the case of Hilbert modular form (with trivial central character):

Corollary 6. *In $Br(F)$ one has*

$$[X] = \mathrm{cor}(\iota_*(\mathrm{res}([c_\alpha])))$$

4. Computing the Brauer Class

Now we have all the ingredients to generalize [8]. The proof is essentially the same. First notice that from theorem 1 (and the assumption $\epsilon = 1$) we know that α^2 is trivial, hence

$$\alpha^2 : G_K \to F^\times/(F^\times)^2$$

is a homomorphism. Let N be the finite Galois extension of K associated with its kernel, i.e. $\ker(\alpha^2) = G_N$. Since $\mathrm{Gal}(N/K) \simeq \mathrm{Im}(\alpha^2) \subset F^\times/(F^\times)^2$ is a 2-torsion group, one has $\mathrm{Gal}(N/K) \simeq (\mathbb{Z}/2\mathbb{Z})^m$ for some positive integer m. Therefore, $N = K(\sqrt{t_1}, ..., \sqrt{t_m})$ for some $t_i \in K$ and if one defines $\sigma_i \in \mathrm{Gal}(N/K)$ with the relations

$$\sigma_i(\sqrt{t_j}) = (-1)^{\delta_{i,j}}\sqrt{t_j}$$

then $\sigma_1, ..., \sigma_m$ form an $\mathbb{F}_2$-basis for $\mathrm{Gal}(N/K)$.

Lemma 5. *In* $\mathrm{Br}(FK)$ *on has:*

$$\iota_*(\mathrm{res}([c_\alpha])) = (t_1, \alpha(\sigma_1)^2)(t_2, \alpha(\sigma_2)^2) \cdots (t_m, \alpha(\sigma_m)^2)$$

where $(a, b) = (a, b)_{FK}$ *denotes the Hilbert symbol.*

Proof. First notice that since $\alpha(\sigma)\sigma(\alpha(\tau))\alpha(\sigma\tau)^{-1}$ is a coboundary, the 2-cocycle $[c_\alpha]$ is also given by the formula $(\sigma, \tau) \mapsto \frac{\alpha(\tau)}{\sigma(\alpha(\tau))}$.

For each $\tau \in G_K$ let

$$\tau(\sqrt{t_i}) = (-1)^{x_i(\tau)}\sqrt{t_i}$$

Then $x_i : G_K \to \mathbb{Z}/2\mathbb{Z}$ is clearly a group homomorphism. Similarly let $y_i : G_{FK} \to \mathbb{Z}/2\mathbb{Z}$ be the homomorphism given by

$$\sigma(\alpha(\sigma_i)) = (-1)^{y_i(\sigma)}\alpha(\sigma_i)$$

for $\sigma \in G_{FK}$. Now since $\{\sigma_i\}_{i=1}^m$ provides an $\mathbb{F}_2$ basis for $\mathrm{Gal}(N/K)$, every element $\tau \in G_K$ can be written as $\eta \prod_{i=1}^m \sigma_i^{x_i(\tau)}$ where η is in $G_N = \ker(\alpha^2)$. Applying α^2 to both sides one gets

$$\alpha^2(\tau) \equiv \prod_{i=1}^m \alpha^2(\sigma_i)^{x_i(\tau)} \pmod{F^{\times 2}}$$

which implies

$$\alpha(\tau) = \lambda \prod_{i=1}^m \alpha(\sigma_i)^{x_i(\tau)}$$

For some $\lambda \in F^\times$. Now one can use this to give a description of $[c_\alpha]$. Applying $\sigma \in G_{FK}$ to the both sides one has

$$\sigma(\alpha(\tau)) = \lambda \prod_{i=1}^m \sigma(\alpha(\sigma_i))^{x_i(\tau)} = \lambda \prod_{i=1}^m (-1)^{y_i(\sigma)x_i(\tau)}\alpha(\sigma_i)^{x_i(\tau)} = \alpha(\tau) \prod_{i=1}^m (-1)^{y_i(\sigma)x_i(\tau)}$$

which gives the description

$$\prod_{i=1}^m (-1)^{y_i(\sigma)x_i(\tau)}$$

for $\iota_*(\mathrm{res}([c_\alpha]))$ in $Br(FK)$. Now, it is well-known that the 2-cocycle $(-1)^{y_i(\sigma)x_i(\tau)}$ is represented by the Hilbert symbol $(t_i, \alpha^2(\sigma_i))$ so we are done. □

From [7] we know that $\Gamma \simeq \mathrm{Gal}(E/F)$ is the group of inner-twists of the form f. Namely, for each $\sigma \in \mathrm{Gal}(E/F)$ there exist a unique character $\chi_\sigma : G_K \to \mathbb{C}^\times$ such that $\chi_\sigma \otimes f = {}^\sigma f$. This is equivalent to saying that for every finite place $\mathfrak{p}$ of K not dividing N one has

$$\chi_\sigma(\mathrm{Frob}_\mathfrak{p}) \cdot a_\mathfrak{p} = \sigma(a_\mathfrak{p})$$

where $a_\mathfrak{p}$ is the $\mathfrak{p}$'th Fourier coefficient (Hecke eigenvalue) of f.

Lemma 6. *The characters* χ_σ *appearing in the inner-twists are exactly characters of* G_K *that factor through* $\mathrm{Gal}(N/K)$*. In partcular, the number of the inner-twists of* f *is* 2^m*.*

Proof. First, we prove that all χ_σ's are trivial of $G_N = \ker(\alpha^2)$. The Sato-Tate conjecture for Hilbert modular forms is known by [1]. This implies that the set of prime ideal $\mathfrak{p}$ of O_K for which $a_\mathfrak{p} \neq 0$ has density 1. Then by Chebotarev's density theorem the Frobenius elements of these primes are dense in G_K, therefore it's enough to check that χ_σ is trivial on the elements of the form $\mathrm{Frob}_\mathfrak{p} \in G_K$ that are in the kernel of α^2 and $a_\mathfrak{p} \neq 0$.

Now if $a_\mathfrak{p} \neq 0$ then by theorem 1, $\alpha^2(\mathrm{Frob}_\mathfrak{p}) \equiv a_\mathfrak{p}^2$ modulo $F^{\times 2}$. Hence, if $\mathrm{Frob}_\mathfrak{p} \in \ker(\alpha^2)$ then $a_\mathfrak{p} \in F$. This implies that $\chi_\sigma(\mathrm{Frob}_\mathfrak{p}) = 1$ by the definition of an inner-twist. So we are done.

To prove that these are all such characters it's enough to prove that the number of character factoring through $\mathrm{Gal}(N/K)$ is equal to the number of the inner-twists. The group of character factoring through $\mathrm{Gal}(N/K)$ is the dual group of $\mathrm{Gal}(N/K)$ and since this is abelian it has exactly $\frac{1}{2^m}$ elements. Then by Chebotarev's density theorem the density of primes $\mathfrak{p}$ (with $a_\mathfrak{p} \neq 0$) that $\mathrm{Frob}_\mathfrak{p} \in G_N$ or equivalently $a_\mathfrak{p} \in F^\times$ is $\frac{1}{2^m}$.

Now, notice that if (σ, χ_σ) is an inner-twist then by definition $\chi_\sigma(\mathrm{Frob}_\mathfrak{p}) \cdot a_\mathfrak{p} = \sigma(a_\mathfrak{p})$. So all χ_σ's are trivial on $\mathrm{Frob}_\mathfrak{p}$ if and only if $a_\mathfrak{p} \in F$. Also, since $a_\mathfrak{p}^2 \in F$ for all $\mathfrak{p}$, $\chi_\sigma^2 = 1$. By [7, Proposition B.3.3] Γ is a finite 2-torsion abelian group. Hence, $\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^n$ for some n. Clearly, $n \leqslant m$ since χ_σ's factor through $\mathrm{Gal}(N/K)$. Now choose an $\mathbb{F}_2$ basis $\sigma^{(1)}, \cdots, \sigma^{(n)}$ for $\Gamma = \mathrm{Gal}(E/F)$. Let G_M be the intersection of kernel of all χ_σ's which is equal to the intersection of the kernel of all $\chi_{\sigma^{(i)}}$'s. Now by Chebotarev's density theorem $M = N$ because they contain the same Frobenius elements of G_K. Since G_N is the intersection of kernel of $\chi_{\sigma^{(i)}}$'s which are all of order 2, one deduces that $n \geqslant m$. This implies $n = m$ and we are done.

□

By the last lemma, the group of characters χ_σ is the dual group of $\mathrm{Gal}(N/K) \simeq (\mathbb{Z}/2\mathbb{Z})^m$. Recall that $\{\sigma\}_{i=1}^n$ is an $\mathbb{F}_2$ basis for $\mathrm{Gal}(N/K)$ satisfying $\sigma_i(\sqrt{t_j}) = (-1)^{\delta_{i,j}}\sqrt{t_j}$ where $N = K(\sqrt{t_1}, ..., \sqrt{t_m})$. Now let $\sigma^{(1)}, \cdots, \sigma^{(m)}$ be a dual basis for this (so each $\sigma^{(i)}$ appear in an inner-twist), i.e.

$$\sigma^{(j)}(\sigma_i) = (-1)^{\delta_{i,j}}.$$

Notice that the fixed field of $\ker(\sigma^{(j)})$ is just $K(\sqrt{t_j})$.

Recall that we need to apply the corestriction map to get back over F and find a formula for $[X]$ in $\mathrm{Br}(F)$. The following well-known lemma helps us to do that.

Lemma 7 ([11], Exercise XIV.2.4)**.** *Let L/F be a finite separable extension and let $\mathrm{cor} : \mathrm{Br}(L) \to \mathrm{Br}(F)$ be the corestriction map. Then for any $a \in L^\times$ and $b \in F^\times$ one has*

$$\mathrm{cor}(a, b)_L = (N_{L/F}(a), b)_F$$

Now we can finally state and prove our main theorem. Note that for any finite place $\mathfrak{p}$ away from N one has $a_\mathfrak{p}^2 \in F$ by proposition 1.

Theorem 2. *Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ be a set of prime ideals of O_K not dividing N and with $a_{\mathfrak{p}_i} \neq 0$ such that $\sigma_i = \mathrm{Frob}_{\mathfrak{p}_i}$ in $\mathrm{Gal}(N/K)$ (such primes exist by Chebotarev's theorem). Then In $\mathrm{Br}(F)$ one has:*

$$[X] = (N_{FK/F}(t_1), a_{\mathfrak{p}_1}^2)(N_{FK/F}(t_2), a_{\mathfrak{p}_2}^2) \cdots (N_{FK/F}(t_m), a_{\mathfrak{p}_m}^2)$$

where $(a,b) = (a,b)_F$ *denotes the Hilbert symbol.*

Proof. Using lemma 5 one only needs to notice that $\alpha(\mathrm{Frob}_{\mathfrak{p}_i})^2 \equiv a_{\mathfrak{p}_i}^2$ modulo $F^{\times 2}$, so they only differ by a square which doesn't affect the Hilbert symbol. Therefore:

$$\iota_*(\mathrm{res}([c_\alpha])) = (t_1, a_{\mathfrak{p}_1}^2)(t_2, a_{\mathfrak{p}_2}^2)\cdots(t_m, a_{\mathfrak{p}_m}^2)$$

Now one applies to corestriction map to both sides. The left hand sides gives us $[X]$ by corollary 6 and the right hand side give us

$$(N_{FK/F}(t_1), a_{\mathfrak{p}_1}^2)(N_{FK/F}(t_2), a_{\mathfrak{p}_2}^2)\cdots(N_{FK/F}(t_m), a_{\mathfrak{p}_m}^2)$$

by the previous lemma, since $a_{\mathfrak{p}_m}^2 \in F^\times$. This proves the statement of the theorem. ☐

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