Weak and strong solutions of Navier-Stokes equations

The equations describing the evolution of incompressible viscous fluid flows are called the Navier-Stokes equations (that were formulated in 1830s +/- 10) and read:

\[
\begin{align*}
\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f \quad \text{in } \Omega \times I, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times I, \\
\nabla \cdot u &= 0 \quad \text{in } \Gamma \times I, \\
\nu \cdot u &= 0 \quad \text{on } \Gamma \times I, \\
u(\cdot, 0) &= u^0 \quad \text{in } \Omega,
\end{align*}
\]

where \( u = (u_x, u_y, u_z) \), \( \nu = \partial_t u \), \( \nu \) = viscosity, \( p \) = pressure, \( \Omega \) = domain in \( \mathbb{R}^3 \), \( \Gamma \) = boundary, \( I = (0, T) \) and \( u_0 \) = initial value.

For a given data, it was proven by Leray (1934) that the Navier-Stokes equations have at least a one weak solution. However, it is still not clear, whether the weak solution is unique or not.

Now by a weak solution we mean a solution that satisfies the above partial differential equations on average, but not necessary on a pointwise manner. For example, the derivatives may not exist in certain points.

On the other hand, a strong solution is defined to satisfy the equation everywhere and in each point.

Lax-Wendroff theorem:
Lax and Wendroff have shown that: If a numerical solution, \( \tilde{u} \), that has been obtained using a conservative scheme, converges, then it converges towards a weak solution of the analytical problem.

Lax theorem:
If \( \tilde{u} \) has been obtained using a conservative scheme and which is consistent, then \( \tilde{u} \) converges to a weak solution of the analytical problem when \( j \to \infty \), i.e., when the number of grid points used goes to infinity.

For numerical mathematicians, the above-mentioned theorems implies the following:

- Different conservative numerical methods may yield different solutions, if the number of grid points is relatively small.

- Different conservative numerical methods that are numerically stable and consistent must converge to the same weak solution if the number of grid points goes to infinity.

- You cannot claim to have found a solution for the physical problem, unless you carried out the calculations with sufficiently large number of grid points, beyond which doubling the number of grid points yield no noticeable improvement.
Taylor expansion:

Let \( f(x) \) that be an infinitely differentiable real function.

An any point \( x \) in the neighbourhood of \( x=x_0 \), the function \( f(x) \) can be represented by a power series of the following form:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \frac{f'''(x_0)}{3!} (x - x_0)^3 \ldots
\]

where \( n! \) stands for the factorial of \( n \) and \( f^{(n)}(x_0) \) for the \( n \)-derivative of \( f \) at \( x = x_0 \).

Examples:

- The function \( f(x) = e^x \) in the neighborhood of \( x=0 \) has the following Taylor series:
  \[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots
\]

- Show that the TS of \( f(x) = \sin(x) \) around \( x = 0 \) is:
  \[
  \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
  \]

- What is the Taylor expansion (TS) of \( f(x) = \frac{e^x}{\sin(x)} \) around \( x=0 \)?
  
  Let \( \frac{e^x}{\sin(x)} \) has an expansion of the form:
  \[
  \frac{e^x}{\sin(x)} = c_1 + c_2 x + c_3 x^2 + c_3 x^3 \ldots
  \]
  
  \[
  \Rightarrow e^x = \left[ c_1 + c_2 x + c_3 x^2 + c_3 x^3 \ldots \right] \times \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right]
  \]
Interpolation techniques

By **interpolation** we mean a procedure for estimating the function at intermediate values.

By **extrapolation** we mean a procedure for estimating the function $f(x)$ at an exterior new value:

- **Linear interpolation:**

$$y(x_2) - y(x_1) = \frac{y(x) - y(x_1)}{x - x_1}$$

$$\Leftrightarrow \quad y(x_2) - y(x_1) = \left[ \frac{x_2 - x_1}{x - x_1} \right] (y(x) - y(x_1))$$

$$\Rightarrow \quad y(x_2) = y(x_1) + \left[ \frac{x_2 - x_1}{x - x_1} \right] (y(x) - y(x_1))$$
3) **Polynomial interpolation:**

Assume we are given \( n+1 \) points:

\[
(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)
\]

We want to construct a polynomial \( p(x) \):

\[
p(x) = y(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots
\]

This polynomial should satisfy the following conditions:

\[
\begin{align*}
    a_n x_0^n + a_{n-1} x_0^{n-1} + \cdots &= y_0 \\
    a_n x_1^n + a_{n-1} x_1^{n-1} + \cdots &= y_1 \\
    \vdots \\
    a_n x_n^n + a_{n-1} x_n^{n-1} + \cdots &= y_n
\end{align*}
\]

\[
\begin{bmatrix}
    x_0^n & x_0^{n-1} & \ldots & x_0 & 1 \\
    x_1^n & x_1^{n-1} & \ldots & x_1 & 1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_n^n & x_n^{n-1} & \ldots & x_n & 1
\end{bmatrix}
\begin{bmatrix}
    a_n \\
    a_{n-1} \\
    \vdots \\
    a_0
\end{bmatrix}
= 
\begin{bmatrix}
    y_0 \\
    y_1 \\
    \vdots \\
    y_n
\end{bmatrix}
\]}
Given the following three points: 

\[(x_0, y_0) = (1, 2)\]
\[(x_1, y_1) = (2, 3)\]
\[(x_2, y_2) = (3, 1)\]

find the best possible approximation for \(y(x = 3/2)\):

\[y(x) = p(x) = a_2 x^2 + a_1 x + a_0\]

\[\Rightarrow \begin{cases} a_2 x_0^2 + a_1 x_0 + a_0 = y_0 \\ a_2 x_1^2 + a_1 x_1 + a_0 = y_1 \\ a_2 x_2^2 + a_1 x_2 + a_0 = y_2 \end{cases} \Rightarrow \begin{cases} a_2 + a_1 + a_0 = 2 \\ a_2 (2^2) + a_1 (2) + a_0 = 3 \\ a_2 (3^2) + a_1 (3) + a_0 = 1 \end{cases}

\[\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}

(+) Check if \(a_0 = 2, a_1 = \frac{-11}{2}, a_2 = -\frac{3}{2}\)

and \(p(x) = y(x) = -\frac{3}{2} x^2 + \frac{11}{2} x - 2\)

is the best possible polynomial that goes through these 3 points?
Lagrangian interpolation:

Given are $n$-points:

Provide an approximation for $y(x)$ that takes into account the information about the $n$-points?

Lagrange answer: The value of $y(x)$ is the linear combination:

$$y(x) = L(x) = \sum_{n=0}^{y_n x_n} y_n x_n,$$

where $y_n = y(x_n)$, $y_0(x_0) = y_0$, $L_n(x) = \prod_{n \neq j} \frac{x-x_j}{x_n-x_j}$

Example:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_n$</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

$$L_0 = \frac{(x-x_3)(x-x_4)}{(x_0-x_3)(x_0-x_4)} = \frac{1}{6} (x-2)(x-3)(x-4)$$

$$L_1 = \frac{(x-x_0)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_3)(x_1-x_4)} = \frac{1}{2} (x-1)(x-3)(x-4)$$

$$L_2 = \frac{(x-x_0)(x-x_0)(x-x_4)}{(x_2-x_0)(x_2-x_3)(x_2-x_4)} = \frac{1}{2} (x-1)(x-2)(x-4)$$

$$L_3 = \frac{(x-x_0)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_2)(x_3-x_4)} = \frac{1}{6} (x-1)(x-2)(x-3)$$

$$y(x) = L(x) = \left[8 \cdot \left(\frac{1}{6}\right)(x-2)(x-3)(x-4)\right] + \left[7 \cdot \left(\frac{1}{2}\right)(x-1)(x-3)(x-4)\right]$$

$$+ \left[6 \cdot \left(\frac{1}{2}\right)(x-1)(x-2)(x-4)\right] + \left[5 \cdot \left(\frac{1}{6}\right)(x-1)(x-2)(x-3)\right]$$
Different finite difference schemes:

How to approximate \( \frac{\partial u}{\partial x} \) at \( u_j \):

\[
\begin{align*}
\text{if } u > 0, \quad u_j & \quad u_j \quad u_j - u_j - 1 \\
& \quad x_j - x_{j-1} \\
\end{align*}
\]

Forward difference scheme:

\[
\frac{\partial u}{\partial x} \approx \frac{u(x_j) - u(x_{j-1})}{x_j - x_{j-1}} = \frac{u_j - u_{j-1}}{h}.
\]

But how accurate is this approximation?

Answer: Taylor-expand \( u_{j-1} \) around \( u_j \):

\[
\begin{align*}
\quad u_{j-1} &= u(x_{j-1}) = u(x_j - h) = u(x_j) - hu_j + \frac{h^2}{2} u_j'' + O(h^3) \\
\Rightarrow \quad \frac{u_j - u_{j-1}}{h} &= \frac{hu_j'' + \frac{h^2}{2} u_j'' + O(h^3)}{h} = u_j'' + O(h)
\end{align*}
\]

Conclusion: The error is of order \( h \), therefore the scheme/approximation is only first order accurate.
**Backward difference:**

\[ \frac{\partial u}{\partial x} \rightarrow \frac{u_{j+1} - u_{j}}{x_{j+1} - x_{j}} = \frac{u_{j+1} - u_{j}}{h} \cdot \]

**Error:** Expand \( u_{j+1} \) around \( u_j \).

\[ = \quad u_{j+1} = u(x_j + h) = u(x_j) + h u' + \frac{1}{2} h^2 u'' + o(h^3) \]

\[ \Rightarrow \quad \frac{u_{j+1} - u_j}{h} = \frac{u_j + h u' + \frac{1}{2} h^2 u'' - u_j}{h} = u' + h u'' = u' + o(h) \]

\[ \Rightarrow \quad \text{The scheme is first order accurate.} \]

**Central finite difference**

\[ \frac{\partial u}{\partial x} \rightarrow \frac{u_{j+1} - u_{j-1}}{2h}. \]

What is the discretization error?

\[ u_{j+1} = u_j + h u' + \frac{1}{2} h^2 u'' + o(h^3) \]

\[ u_{j-1} = u_j - h u' + \frac{1}{2} h^2 u'' - o(h^3) \]

\[ \Rightarrow \quad \frac{u_{j+1} - u_{j-1}}{2h} = \left[ u_j + h u' + \frac{1}{2} h^2 u'' + o(h^3) \right] - \left[ u_j - h u' + \frac{1}{2} h^2 u'' - o(h^3) \right] \]

\[ = \quad \frac{2h u'}{2h} + \frac{o(h^3)}{h} = u' + o(h) \]

Thus the scheme is of second order accuracy.
Higher order derivatives:

\[
\frac{\partial^2 T}{\partial x^2} = \frac{3}{8} \left( \frac{\partial T}{\partial x} \right)
\]

Let \( \Phi = \frac{\partial T}{\partial x} \).

\[\Rightarrow \frac{\partial^2 T}{\partial x^2} \rightarrow \frac{\Delta \Phi}{\Delta x} = \frac{T_j - T_{j-1}}{h} \]

\[T_j = \left. \frac{\Delta T}{\Delta x} \right|_{x_j} = \frac{T_{j+1} - T_j}{h} \]

\[T_{j-1} = \left. \frac{\Delta T}{\Delta x} \right|_{x_{j-1}} = \frac{T_j - T_{j-1}}{h} \]

\[\Rightarrow \frac{\Delta \Phi}{\Delta x} = \frac{1}{h} \left[ \frac{T_{j+1} - T_j}{h} \right] - \frac{1}{h} \left[ \frac{T_j - T_{j-1}}{h} \right] \]

\[= \frac{1}{h^2} \left[ T_{j+1} - 2T_j + T_{j-1} \right] \]

**Accuracy?** Taylor-expand \( T_{j+1} \) around \( T_j \).

\[T_{j+1} = T_j + h T' + \frac{h^2}{2} T'' + O(h^3) + O(h^4) \]

\[T_{j-1} = T_j - h T' + \frac{h^2}{2} T'' - O(h^3) + O(h^4) \]

\[\Rightarrow \frac{T_{j+1} - 2T_j + T_{j-1}}{h^2} = \frac{2T_j + h^2 T'' + O(h^4) - 2T_j}{h^2} = T'' + O(h^2) \Rightarrow \text{This discretization scheme is second order accurate.} \]
Taylor series expansion

\[
\frac{\partial T}{\partial t} = \mathcal{O}\left(\frac{\partial T}{\partial t}\right) + \mathcal{O}\left(\frac{\partial^2 T}{\partial t^2}\right) + \mathcal{O}\left(\frac{\partial^3 T}{\partial t^3}\right) + \cdots
\]

\[
\frac{\partial T}{\partial t} = \mathcal{O}\left(\frac{\partial T}{\partial x}\right) + \mathcal{O}\left(\frac{\partial^2 T}{\partial x^2}\right) + \mathcal{O}\left(\frac{\partial^3 T}{\partial x^3}\right) + \cdots
\]

\[
\frac{\partial T}{\partial x} = \mathcal{O}\left(\frac{\partial^2 T}{\partial x^2}\right) + \mathcal{O}\left(\frac{\partial^3 T}{\partial x^3}\right) + \cdots
\]

\[
L_a(T) = \frac{1}{\Delta t} \left\{ \overline{T}_{j+1} - \overline{T}_j - \left( \frac{\alpha \Delta t}{\Delta x^2} \right) \left( \overline{T}_{j+1} - 2\overline{T}_j + \overline{T}_{j-1} \right) \right\}
\]

\[
= \frac{1}{\Delta t} \left\{ \left( \frac{\Delta t}{\partial t} \right) + \left( \frac{\Delta t}{\partial x} \right)^2 + \left( \frac{\Delta t}{\partial x} \right)^3 + \cdots \right\}
\]

\[
-2 \left( \frac{\alpha \Delta t}{\Delta x^2} \right) \left\{ \left( \frac{\Delta x}{\partial x} \right)^2 + \left( \frac{\Delta x}{\partial x} \right)^4 + \left( \frac{\Delta x}{\partial x} \right)^6 + \cdots \right\}
\]

\[
L_a(T) = \frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial t^2} + \mathcal{O}(\delta t) + \mathcal{O}(\Delta x^2)
\]

Truncation error
The one-dimensional heat equation:

Consider the following heat diffusion equation:

\[ \frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2} \]

In the absence of transport, the heat diffuses in symmetric manner, provided the diffusion coefficient, \( \chi \), and the BC are symmetric too. If advection (transport) is included, then this symmetry will be broken as can be seen from the following two figures.

Assume that both edges of the metal rod are kept at certain constant temperatures \( T_u \) and \( T_d \). Let the rod be heated at the center for a certain period of time. How do the initial, intermediate and final profiles of the temperature look like?

**Intuitively (without performing analytical or numerical calculation) we may conclude that**: the BCs and the ICs play an essential role in determining the form and evolution of the solution at any time \( t \).

There two types of problems: **Initial value and boundary value problems**. The strength of dependence on the ICs and BCs determine the type of the problem.
Numerically:

The temperature depends on t as well as on x, i.e. \( T = T(t,x) \). Thus for each value of “x” and value of “t” there is a suitable value, and hopefully a unique value, for \( T \).

An explicit discretization of this equation yields the following form:

\[
\frac{T_{j}^{n+1} - T_{j}^{n}}{\delta t} = \frac{\chi}{\Delta x} \left( \frac{T_{j+1}^{n} - T_{j}^{n}}{\Delta x} - \frac{T_{j}^{n} - T_{j-1}^{n}}{\Delta x} \right)
\]

\( \iff T_{j}^{n+1} = sT_{j-1}^{n} + (1 - 2s)T_{j}^{n} + sT_{j+1}^{n} \),

where \( s = \frac{\delta t \times \chi}{\Delta x^2} \).
The Von Neumann stability analysis shows that this method is numerically stable, if the Courant-Friedrich-Levy number, or shortly CFL < 1, where \( \text{CFL} = \frac{\delta t \times \chi}{\Delta x^2} \).

QL56_2: Solve the heat equation in the domain \( D = [t] \times [x] = [0,1] \times [0,1] \) using the explicit formulation for the following parameters: \( \chi = 1 \), \( N_s \) (Number of grid points in x-direction) = 100, \( \text{ans } s = 0.1, 0.2, 0.4, 0.8, 1.2 \). The IC and BC read: \( T(t=0) = 1, T(t,0) = 2, T(t,1)=2 \). Plot the solutions at times: \( t=0.1, 0.2, 0.4 \) and 1.0.

---

### Table 9.1. Algebraic (discretised) schemes for the convection equation \( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0 \)

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Algebraic form</th>
<th>Truncation error ( (E) ) (leading terms)</th>
<th>Amplification factor ( G(\theta = \text{max},dx) )</th>
<th>Stability restrictions</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>FTCS</td>
<td>( \frac{\Delta T_{i+1}^n}{\Delta t} + u \Delta x \Delta T_{i+1}^n = 0 )</td>
<td>( Cu \frac{\Delta x^2}{6} \frac{\partial^3 T}{\partial x^3} )</td>
<td>( 1 - i \sin \theta )</td>
<td>unstable</td>
<td>( C = \frac{\Delta t}{\Delta x} )</td>
</tr>
<tr>
<td>Upwind</td>
<td>( \frac{\Delta T_{i+1}^n}{\Delta t} + \frac{T_{i+1}^n - T_{i-1}^n}{\Delta x} = 0 )</td>
<td>( \frac{\Delta x^2}{6} \frac{\partial^3 T}{\partial x^3} )</td>
<td>( 1 - C \frac{\Delta x}{\Delta t} \sqrt{\frac{\partial^3 T}{\partial x^3}} )</td>
<td>( C \leq 1 )</td>
<td>( \Delta \Delta T_{i+1}^n = T_{i+1}^n - T_{i}^n )</td>
</tr>
<tr>
<td>Leapfrog</td>
<td>( \frac{T_{i+1}^n - T_{i-1}^n}{2 \Delta t} + u \Delta x \Delta T_{i+1}^n = 0 )</td>
<td>( \frac{\Delta t^2}{6} \frac{\partial^3 T}{\partial x^3} )</td>
<td>( -i \sin \theta \pm (1 - C^2 \sin^2 \theta)^2 )</td>
<td>( C \leq 1 )</td>
<td>( C \leq 1 )</td>
</tr>
</tbody>
</table>

---

**From Fletcher: “Computational techniques ... (1990)”**